CSE 311: Foundations of Computing I

Section : Induction Solutions

1. Induction with Sums

(a) Prove for all $n \in \mathbb{N}$ that if you have two groups of numbers, $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, such that $\forall(i \in [n]).\ a_i \leq b_i$, then it must be that:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i$$

Solution:

We prove this by induction on n:

Base Case ($n = 0$). We know that:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{0} a_i = 0 = \sum_{i=1}^{0} b_i = \sum_{i=1}^{n} b_i$$

So the claim is true for $n = 0$.

Induction Hypothesis. Suppose for some $k \in \mathbb{N}$ that $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$ for all groups of numbers $a_1, \cdots, a_k$ and $b_1, \cdots, b_k$ such that $a_i \leq b_i$ for all $i \in [k]$

Induction Step. Let the groups of numbers $a_1, \cdots, a_{k+1}$ and $b_1, \cdots, b_{k+1}$ be two groups such that $a_i \leq b_i$ for all $i \in [k+1]$.
Note that

$$\begin{aligned}
\sum_{i=0}^{k+1} a_i &= \sum_{i=0}^{k} a_i + a_{k+1} && \text{[Splitting the summation]} \\
&\leq \sum_{i=0}^{k} b_i + a_{k+1} && \text{[By IH]} \\
&\leq \sum_{i=0}^{k} b_i + b_{k+1} && \text{[By Assumption]} \\
&\leq \sum_{i=1}^{k+1} b_i && \text{[Algebra]}
\end{aligned}$$

Thus we have shown that if the claim is true for k, it is true for $k + 1$.

Therefore, we have shown the claim for all $n \in \mathbb{N}$ by induction.

(b) For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = \sum_{i=1}^{n} i^2.$$

For all $n \in \mathbb{N}$, prove that $S_n = \frac{1}{6}n(n+1)(2n+1)$.

Solution:

Let P(n) be the statement "$S_n = \frac{1}{6}n(n+1)(2n+1)$" defined for all $n \in \mathbb{N}$. We prove that P(n) is true for all $n \in \mathbb{N}$ by induction on n.

Base Case. When $n = 0$, we know the sum of the squares of the first n positive integers is the sum of no terms, so we have a sum of 0. Thus, $S_0 = 0$. Since $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$, we know that P$(0)$ is true.

Induction Hypothesis. Suppose that P(k) is true for some $k \in \mathbb{N}$.

Induction Step. Examining S_{k+1}, we see that

$$S_{k+1} = \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the induction hypothesis, we know that $S_k = \frac{1}{6}k(k+1)(2k+1)$. Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned} S_{k+1} &= S_k + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= (k+1)\left(\frac{1}{6}k(2k+1) + (k+1)\right) \\ &= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\ &= \frac{1}{6}(k+1)\left(2k^2+7k+6\right) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \end{aligned}$$

Thus, we can conclude that P$(k+1)$ is true.

Therefore, because the base case and induction step hold, P(n) is true for all $n \in \mathbb{N}$ by induction.

(c) Define the triangle numbers as $\triangle_n = 1+2+\cdots+n$, where $n \in \mathbb{N}$. We showed in lecture that $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} i^3 = \triangle_n^2$$

Solution:

First, note that $\triangle_n = \sum_{i=0}^{n} i$. So, we are trying to prove $\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$.

Let P(n) be the statement:

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$$

We prove that P(n) is true for all $n \in \mathbb{N}$ by induction on n.

Base Case. $0^3 = 0^2$, so P(0) holds.

Induction Hypothesis. Suppose that $\mathsf{P}(k)$ is true for some $k \in \mathbb{N}$.

Induction Step. We show $\mathsf{P}(k+1)$:

$$\begin{aligned}
\sum_{i=0}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 && \text{[Take out a term]} \\
&= \left(\sum_{i=0}^{k} i\right)^2 + (k+1)^3 && \text{[Induction Hypothesis]} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[Substitution from part (a)]} \\
&= (k+1)^2 \left(\frac{k^2}{2^2} + (k+1)\right) && \text{[Factor } (k+1)^2\text{]} \\
&= (k+1)^2 \left(\frac{k^2+4k+4}{4}\right) && \text{[Add via comon denominator]} \\
&= (k+1)^2 \left(\frac{(k+2)^2}{4}\right) && \text{[Factor numerator]} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 && \text{[Take out the square]} \\
&= \left(\sum_{i=0}^{k+1} i\right)^2 && \text{[Substitution from part (a)]}
\end{aligned}$$

Therefore, $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction.

2. Induction

(a) Prove that $9 \mid n^3 + (n+1)^3 + (n+2)^3$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Induction Hypothesis: Assume that $9 \mid j^3 + (j+1)^3 + (j+2)^3$ for an arbitrary integer $j > 1$. Note that this is equivalent to assuming that $j^3 + (j+1)^3 + (j+2)^3 = 9k$ for some integer k.

Induction Step: Goal: Show $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$

$$\begin{aligned}
(j+1)^3 + (j+2)^3 + (j+3)^3 &= (j+3)^3 + 9k - j^3 \text{ for some integer } k \quad \text{[Induction Hypothesis]} \\
&= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\
&= 9j^2 + 27j + 27 + 9k \\
&= 9(j^2 + 3j + 3 + k)
\end{aligned}$$

So $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$, so $P(j) \to P(j+1)$ for an arbitrary integer $j > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

(b) Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Solution:

Let $P(n)$ be "$6n+6 < 2^n$". We will prove $P(n)$ for all integers $n \geq 6$ by induction.

Base Case $(n = 6)$**:** $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

Induction Hypothesis: Assume that $6j + 6 < 2^j$ for an arbitrary integer $j \geq 6$.

Induction Step: Goal: Show $6(j+1) + 6 < 2^{j+1}$

$$\begin{aligned}
6(j+1) + 6 &= 6j + 6 + 6 \\
&< 2^j + 6 && \text{[Induction Hypothesis]} \\
&< 2^j + 2^j && \text{[Since } 2^j > 6 \text{, since } j \geq 6] \\
&< 2 \cdot 2^j \\
&< 2^{j+1}
\end{aligned}$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \geq 6$.

Conclusion: $P(n)$ holds for all integers $n \geq 6$ by induction.

(c) Define

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$.

Solution:

We define H_i more formally as $\sum_{k=1}^{i} \frac{1}{k}$. Let $P(n)$ be "$H_{2^n} \geq 1 + \frac{n}{2}$". We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case $(n = 0)$**:** $H_{2^0} = H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$, so $P(0)$ holds.

Induction Hypothesis: Assume that $H_{2^j} \geq 1 + \frac{j}{2}$ for an arbitrary integer $j \in \mathbb{N}$.

Induction Step: Goal: Show $H_{2^{j+1}} \geq 1 + \dfrac{j+1}{2}$

$$\begin{aligned}
H_{2^{j+1}} &= \sum_{k=1}^{2^{j+1}} \frac{1}{k} \\
&= \sum_{k=1}^{2^j} \frac{1}{k} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \\
&\geq 1 + \frac{j}{2} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} && \text{[Induction Hypothesis]} \\
&\geq 1 + \frac{j}{2} + 2^j \cdot \frac{1}{2^{j+1}} && \text{[There are } 2^j \text{ terms in } [2^j+1, 2^{j+1}] \text{ and each is at least } \tfrac{1}{2^{j+1}}] \\
&\geq 1 + \frac{j}{2} + \frac{2^j}{2^{j+1}} \\
&\geq 1 + \frac{j}{2} + \frac{1}{2} \geq 1 + \frac{j+1}{2}
\end{aligned}$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \in \mathbb{N}$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.