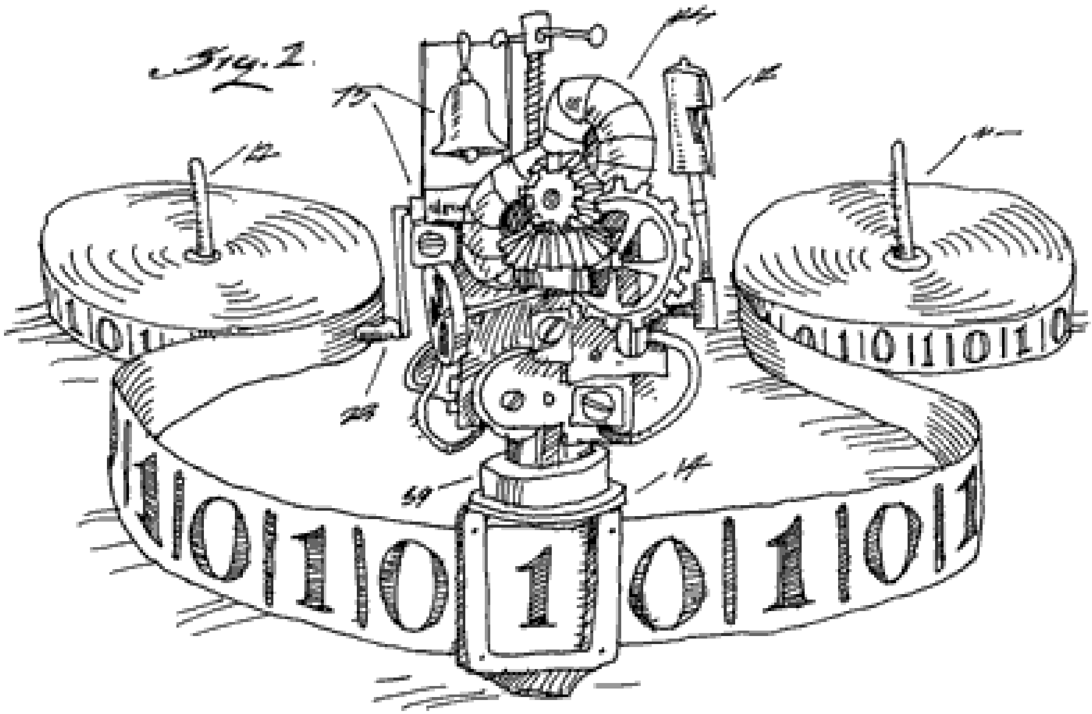
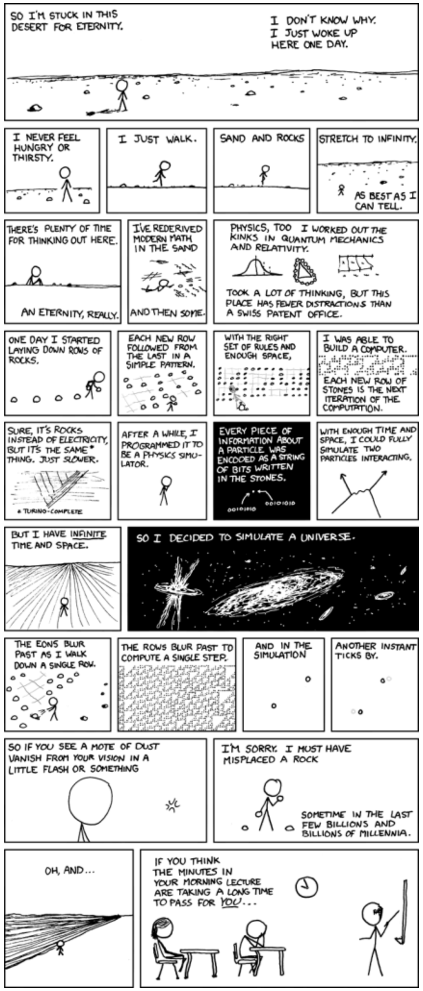
Spring 2015

Lecture 28: The halting problem and undecidability





We saw that the real numbers between 0 and 1 are **uncountable**.

Suppose, for the sake of contradiction, that there is a list of them:

		1	2	3	4						
r_1	0.	5^{1}	0	0	0						
r_2	0.	3	3^{5}	3	3						
r_3	0.	1	4	2^{5}	8	5	7	1	4	...	...
r_4	0.	1	4	1	5^{1}	9	2	6	5	...	...
						2^{5}	1	2	2	...	...
						0	0^{5}	0	0	...	...
						8	1	8^{5}	2	...	...

Flipping rule:
If digit is 5, make it 1.
If digit is not 5, make it 5.

For every $n \geq 1$:
$r_n \neq 0.\tilde{x}_{11}\tilde{x}_{22}\tilde{x}_{33}\tilde{x}_{44}\tilde{x}_{55}\cdots$
because the numbers differ on the nth digit!

So the list is incomplete, which is a contradiction.

Thus the real numbers between 0 and 1 are **uncountable**.

the set of all functions $f : \mathbb{N}^{+} \to \{0, \dots, 9\}$ is uncountable

Supposed listing of all the functions:

	1	2	3	4	5	6	7	8	9	...
f_1	5	0	0	0	0	0	0	0	...	...
f_2	3	3	3	3	3	3	3	3	...	...
f_3	1	4	2	8	5	7	1	4	...	...
f_4	1	4	1	5	9	2	6	5	...	...
f_5	1	2	1	2	2	1	2	2	...	...
f_6	2	5	0	0	0	0	0	0	...	...
f_7	7	1	8	2	8	1	8	2	...	...
f_8	6	1	8	0	3	3	9	4	...	...
...	...	....	....	...	...	...	...	...	...	

the set of all functions $f : \mathbb{N} \to \{0, \dots, 9\}$ is uncountable

Supposed listing of all the functions:

	1	2	3	4						
f_1	5^{1}	0	0	0						
f_2	3	3^{5}	3	3						
f_3	1	4	2^{5}	8	5	7	1	4	...	...
f_4	1	4	1	5^{1}	9	2	6	5	...	...
f_5	1	2	1	2	2^{5}	1	2	2	...	...
f_6	2	5	0	0	0	0^{5}	0	0	...	...
f_7	7	1	8	2	8	1	8^{5}	2	...	...
f_8	6	1	8	0	3	3	9	4^{5}	...	...
...	...	....	....	...	...	...	...	...	...	

Flipping rule:

If $f_n(n) = 5$, set $D(n) = 1$

If $f_n(n) \neq 5$, set $D(n) = 5$

the set of all functions $f : \mathbb{N} \to \{0, \dots, 9\}$ is uncountable

Supposed listing of all the functions:

	1	2	3	4						
f_1	5^{1}	0	0	0						
f_2	3	3^{5}	3	3						
f_3	1	4	2^{5}	8	5	7	1	4	...	...
f_4	1	4	1	5^{1}	9	2	6	5	...	...
f_5	1	2	1	2	2^{5}	1	2	2	...	...
f_6	2	5	0	0	0	0^{5}	0	0	...	...
f_7	7	1	8	2	8	1	8^{5}	2	...	...

Flipping rule:

If $f_n(n) = 5$, set $D(n) = 1$

If $f_n(n) \neq 5$, set $D(n) = 5$

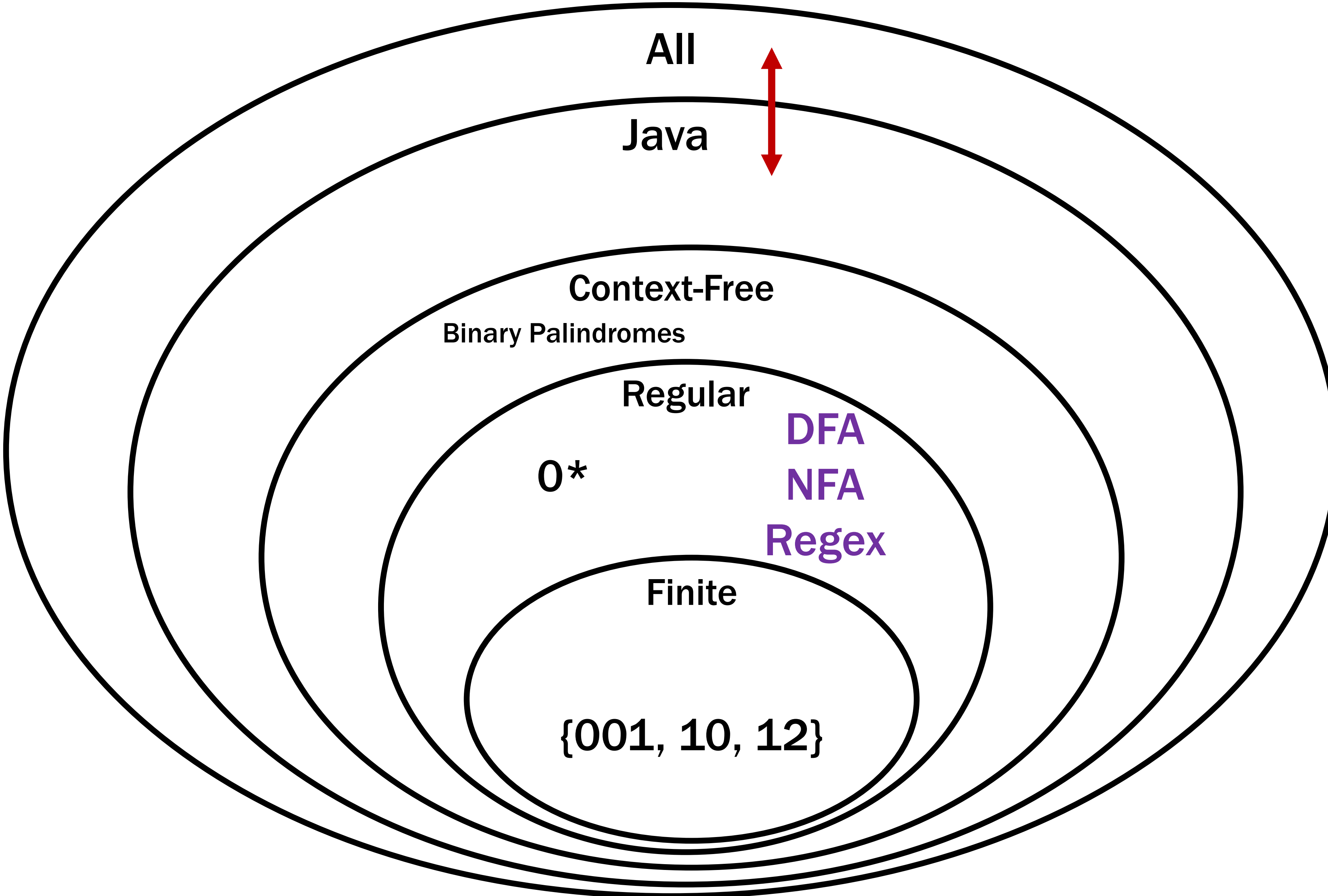
For all n, we have $D(n) \neq f_n(n)$. Therefore $D \neq f_n$ for any n and the list is incomplete! $\Rightarrow$ $\{f \mid f : \mathbb{N} \to \{0,1, \dots, 9\}\}$ is **not** countable

We have seen that:

- [last time] The set of all (Java) programs is countable
- The set of all functions $f : \mathbb{N} \to \{0, \ldots, 9\}$ is not countable

So: There must be some function $f : \mathbb{N} \to \{0, \ldots, 9\}$ that is not computable by any program!

recall our language picture



a cse 141 assignment

Students should write a Java program that:

- Prints “Hello” to the console
- Eventually exits

GradeIt, PracticeIt, etc. need to grade the students.

How do we write that grading program?

What does this program do?

```
_(__,___,____){___/__<=1?_(__,___+1,___
_):!(___%__)?_(__,___+1,0):___%__==___ /
__&&!____?(printf("%d\t",___/__),_(__,_
__+1,0)):___%__>1&&___%__<___/__?_( __,1+
___,____+!(___/__%(___%__))):___<__*__
?_(__,___+1,____):0;}main(){_(100,0,0);}
```

follow up question #2

```
public static void collatz(n) {
    if (n == 1) {
        return 1;
    }
    if (n % 2 == 0) {
        return collatz(n/2)
    }
    else {
        return collatz(3n + 1)
    }
}
```

5, 16, 8, 4, 2,
1.

7, 22, 11, 34, 17,
52, 26, 13,
40, 20, 10,
5, 16, 8, 4, 2, 1

What does this program do?

… **on n=5?**

… **on n=10000000000000000001?**

Students should write a Java program that:

- Prints "Hello" to the console
- Eventually exits

GradeIt, PracticeIt, etc. need to grade the students.

How do we write that grading program?

IMPOSSIBLE

We're going to be talking about *Java code*.

CODE(P) will mean "the code of the program P"

So, consider the following function:

```
public String P(String x) {
    return new String(Arrays.sort(x.toCharArray());
}
```

What is P(CODE(P))?

"((()))..;AACPSSaaabceeggghiiiilnnnnnooprrrrrrrrrrrssstttttttuuwxxyy{}"

the Halting problem

Given: - CODE(**P**) for any program **P**

- input **x**

Output: **true** if **P** halts on input **x**

false if **P** does not halt on input **x**

It turns out that it isn't possible to write a program that solves the Halting Problem.

H(CODE(P), x)

proof by contradiction

cannot exist

- Suppose that **H** is a Java program that solves the Halting problem. Then we can write this program:

x = CODE(D)

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true);    /* don't halt */
    }
    else {
        return;          /*    halt    */
    }
}
```

H(CODE(D), CODE(D))

D(CODE(D)) does halt

⇓

D(CODE(D)) does not halt

- Does **D**(CODE(**D**)) halt?

A: D(CODE(D)) does not halt

⟹ D(CODE(D)) halts

Does D(CODE(D)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;       /*   halt   */
    }
}
```

H solves the halting problem implies that

H(CODE(D),x) is **true** iff D(x) halts, H(CODE(D),x) is **false** iff not

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;       /*   halt   */
    }
}
```

Does D(CODE(D)) halt?

H solves the halting problem implies that
H(CODE(D),x) is **true** iff D(x) halts, H(CODE(D),x) is **false** iff not

Suppose D(CODE(D)) **halts**.
Then, we must be in the **second** case of the if.
So, H(CODE(D), CODE(D)) is **false**
Which means D(CODE(D)) **doesn't halt**

Does **D**(CODE(**D**)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;        /*   halt   */
    }
}
```

H solves the halting problem implies that

H(CODE(**D**),x) is **true** iff **D**(x) halts, H(CODE(**D**),x) is **false** iff not

Suppose **D**(CODE(**D**)) **halts**.

Then, we must be in the **second** case of the if.

So, H(CODE(**D**), CODE(**D**)) is **false**

Which means **D**(CODE(**D**)) **doesn't halt**

Suppose **D**(CODE(**D**)) **doesn't halt**.

Then, we must be in the **first** case of the if.

So, H(CODE(**D**), CODE(**D**)) is **true**.

Which means **D**(CODE(**D**)) **halts**.

Does **D**(CODE(**D**)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;       /*   halt   */
    }
}
```

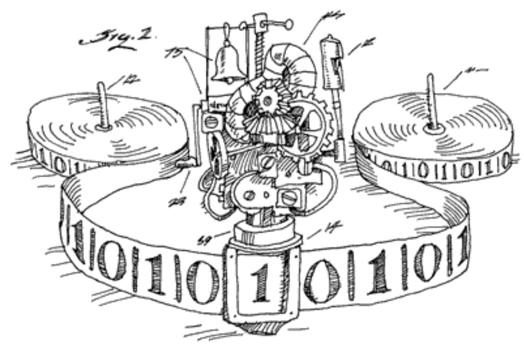
H solves the halting problem implies that
H(CODE(**D**),x) is **true** iff **D**(x) halts, H(CODE(**D**),x) is **false** iff not

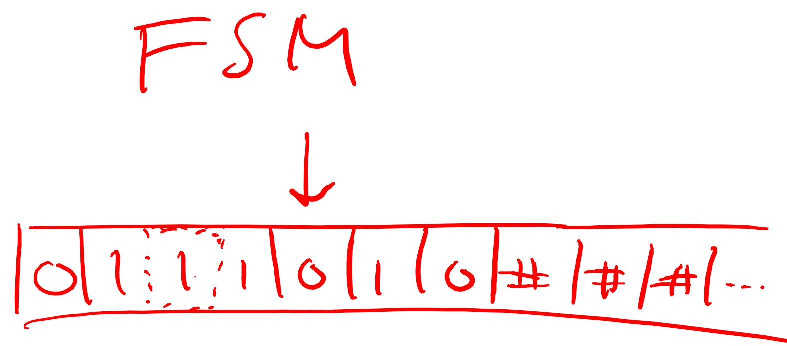
Suppose **D**(CODE(**D**)) **halts**.
Then, we must be in the **second** case of the if.
So, H(CODE(**D**), CODE(**D**)) is **false**
Which means **D**(CODE(**D**)) **doesn't halt**

Suppose **D**(CODE(**D**)) **doesn't halt**.
Then, we must be in the **first** case of the if.
So, H(CODE(**D**), CODE(**D**)) is **true**.
Which means **D**(CODE(**D**)) **halts**.

Contradiction!

- We proved that there is no computer program that can solve the Halting Problem.
  - There was nothing special about Java*   [Church-Turing thesis]





- This tells us that there is no compiler that can check our programs and guarantee to find any infinite loops they might have.

connection to diagonalization

Some possible inputs **x**

programs **P**

	$\langle P_1\rangle$	$\langle P_2\rangle$	$\langle P_3\rangle$	$\langle P_4\rangle$	$\langle P_5\rangle$	$\langle P_6\rangle$	....					
P_1	0	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1	0	1	0	...

P_9 . . . .

. . . .

.

(P,x) entry is **1** if program **P** halts on input **x**
and **0** if it runs forever

connection to diagonalization

Some possible inputs x

programs P

	$\langle P_1\rangle$	$\langle P_2\rangle$	$\langle P_3\rangle$	$\langle P_4\rangle$	$\langle P_5\rangle$	$\langle P_6\rangle$	....					
P_1	0^{1}	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1^{0}	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1^{0}	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0^{1}	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1^{0}	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1^{0}	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0^{1}	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1^{0}	0	1	0	...
P_9	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
.												

(P,x) entry is **1** if program **P** halts on input **x**
and **0** if it runs forever

- Can use undecidability of the halting problem to show that other problems are undecidable.

- For instance:
  $\mathrm{EQUIV}(P, Q)$ = **True** if $P(x) = Q(x)$ for every input x
  False otherwise

$\boxed{Q_0(x) - \text{Halts on } x}$

$\mathrm{EQUIV}(P, Q_0)$

$\{ (m,n) : m, n \in \mathbb{Z} \}$ ← fair game

Rice's theorem

Not *every* problem on programs is undecidable!

Which of these is decidable?

- Input CODE(P) and x
  Output: true if P prints "ERROR" on input x
  after less than 100 steps
  false otherwise

DECIDABLE

- Input CODE(P) and x
  Output: true if P prints "ERROR" on input x
  after more than 100 steps
  false otherwise

UNDECIDABLE

Compilers Suck Theorem (informal):

Any "non-trivial" property the **input-output behavior** of Java programs is undecidable.

foundations I, complete (almost)

What's next?

Foundations II: Probability, statistics, and uncertainty.

The **final exam** is Monday, Jun 8, 2015, 2:30-4:20 p.m. in MLR 301.
Notes: One page of notes allowed, front and back.
Review sessions:
- Saturday, June 6th, 2015: 1pm in EEB 105 (James)
- Sunday, June 7th, 2015: 2pm in EEB 105 (TAs)

And then… summer!

DAY	COND	HIGH	LOW	DESCRIPTION	PRECIP	WIND
TONIGHT Jun 4		--	52°F	Mostly Clear	0%	NNE 7 mph
FRI Jun 5		76°	54°	Sunny	0%	N 10 mph
SAT Jun 6		80°	57°	Mostly Sunny	0%	N 9 mph
SUN Jun 7		82°	57°	Mostly Sunny	0%	NNW 8 mph
MON Jun 8		79°	55°	Sunny	10%	NNW 9 mph
TUE Jun 9		78°	55°	Sunny	0%	NNW 8 mph
WED Jun 10		77°	54°	Sunny	0%	WSW 6 mph
THU Jun 11		77°	56°	Sunny	10%	W 7 mph
FRI Jun 12		79°	56°	Sunny	0%	NW 8 mph
SAT Jun 13		76°	55°	Sunny	0%	WNW 8 mph