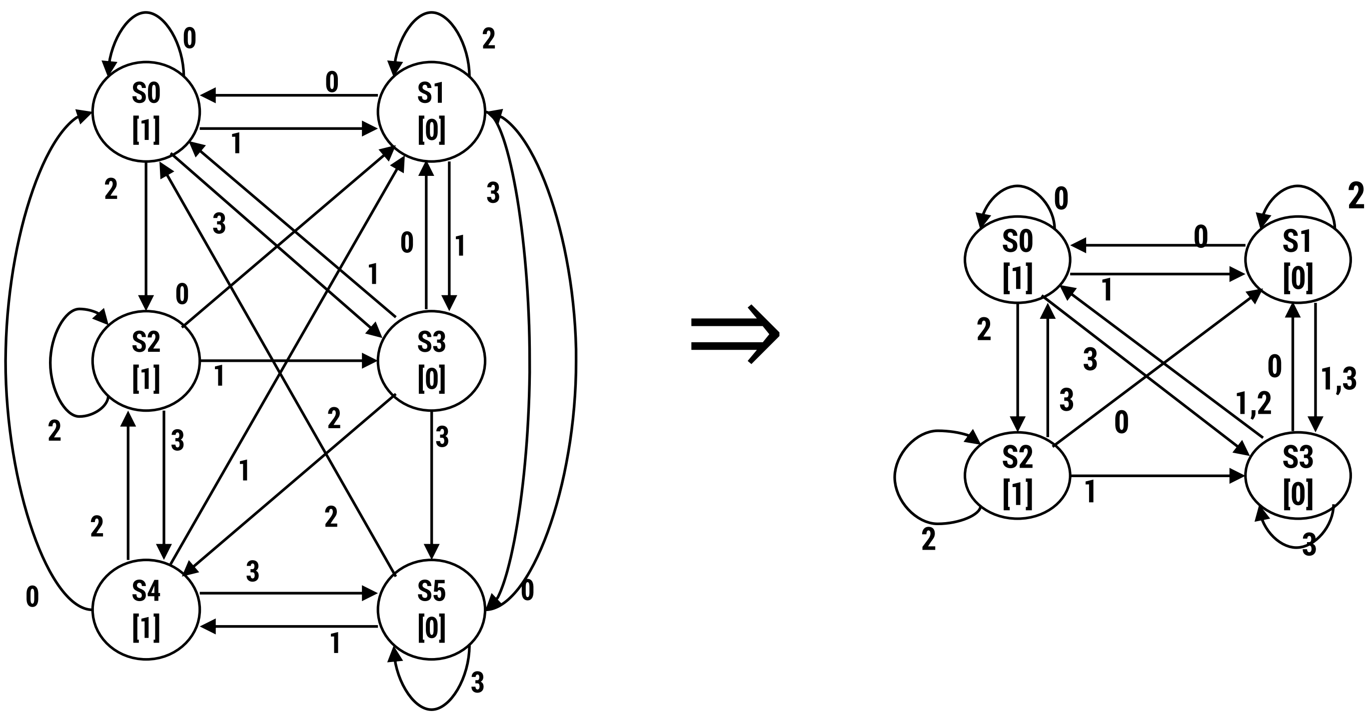
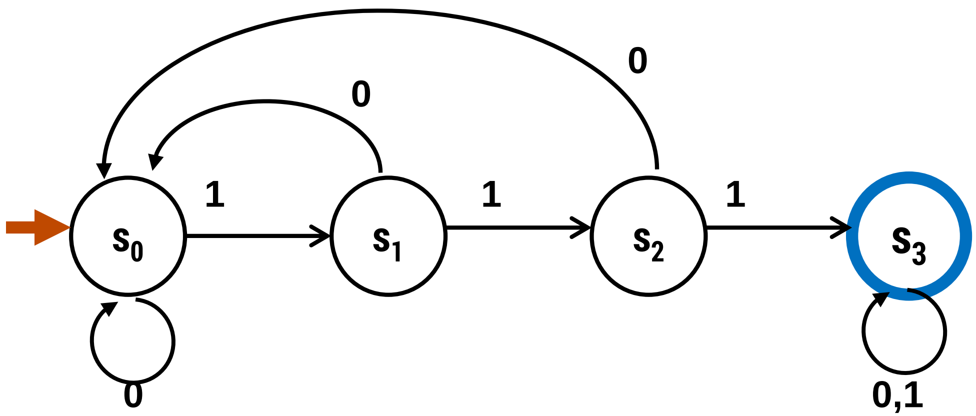
# Spring 2015

## Lecture 24: DFAs, NFAs, and regular expressions

- FSMs with output at states
- State minimization

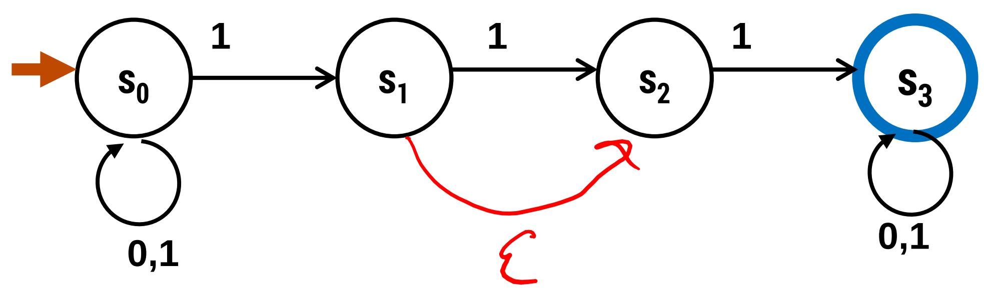


Lemma: The language recognized by a DFA is the set of strings x that label some path from its start state to one of its final states



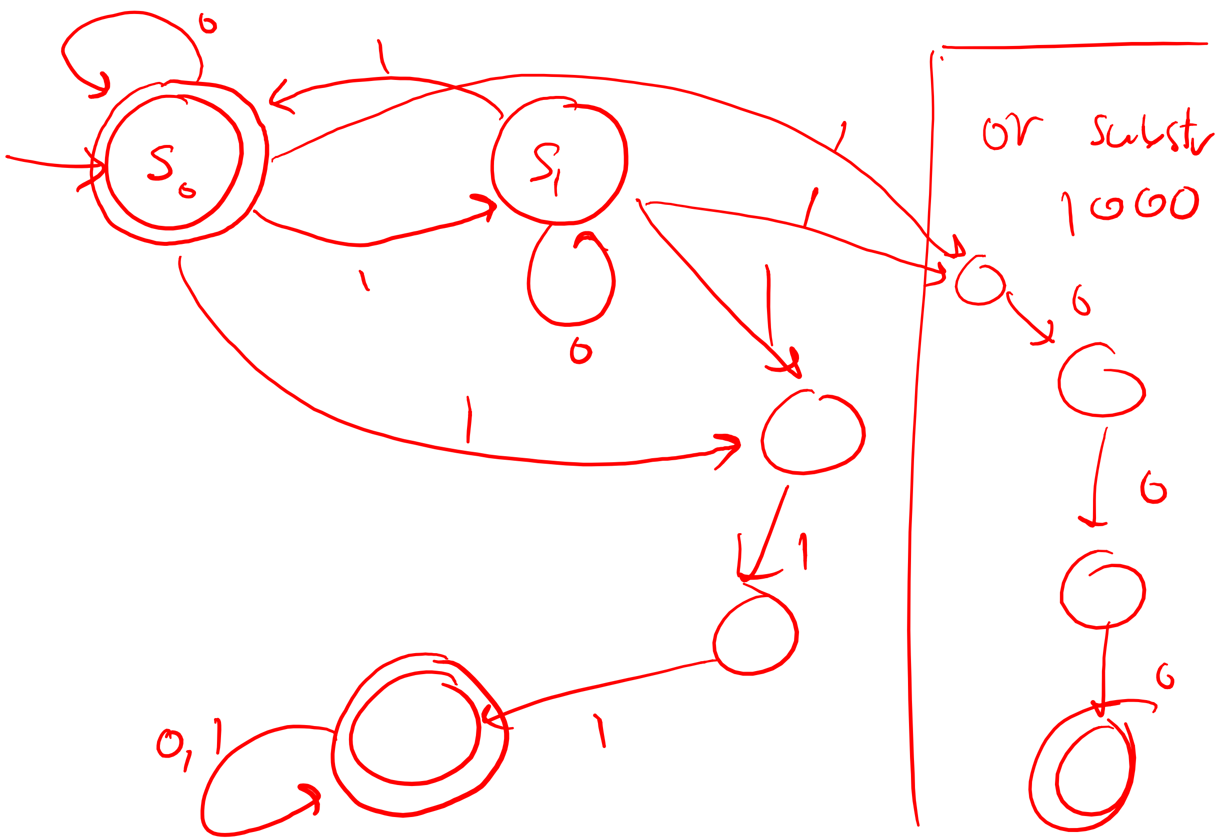
# nondeterministic finite automaton (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol--- can have 0 or >1
  - Also can have edges labeled by empty string ε
- **Definition:** x is in the language recognized by an NFA if and only if x labels a path from the start state to some final state



binary strings that have

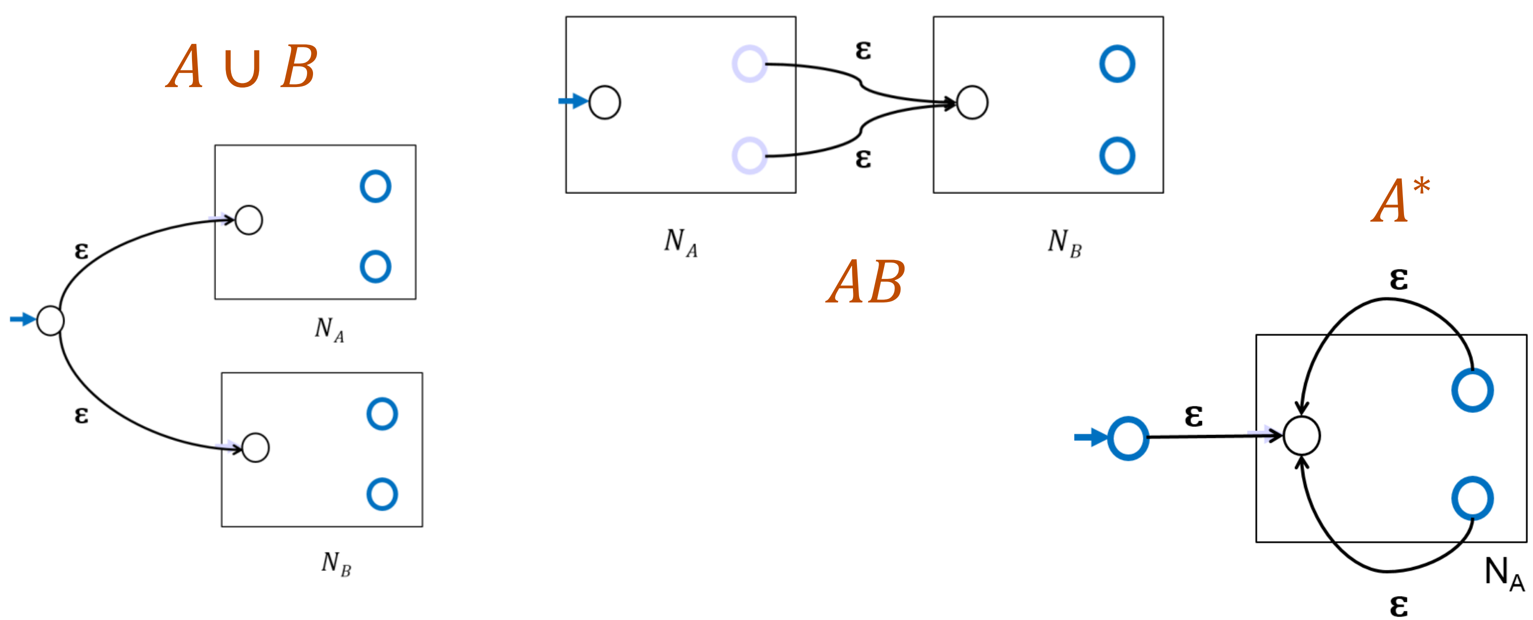
- an even # of 1's
- or contain the substring 111 or 1000



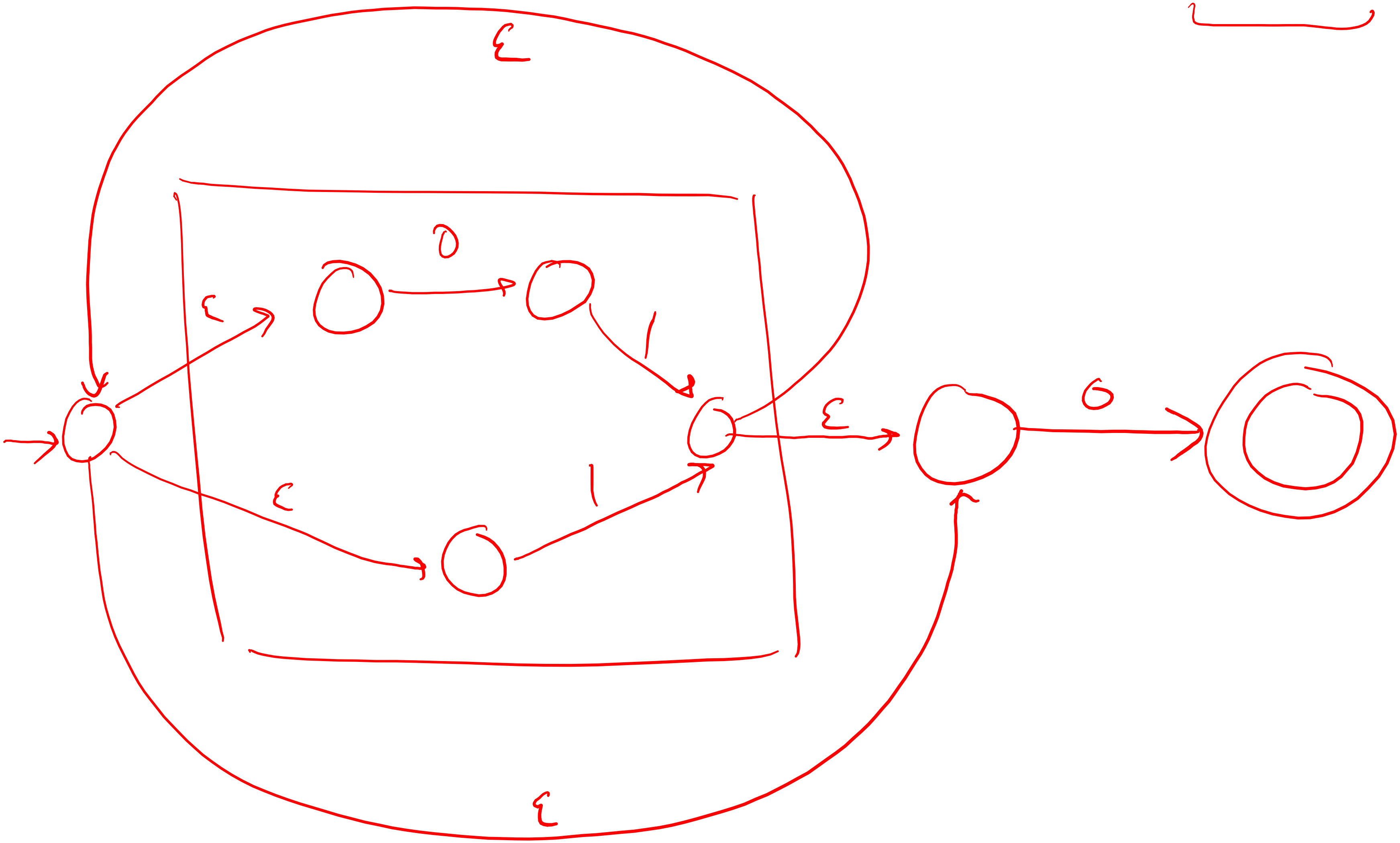
# NFAs and regular expressions

**Theorem:** For any set of strings (language) $A$ described by a regular expression, there is an NFA that recognizes $A$.

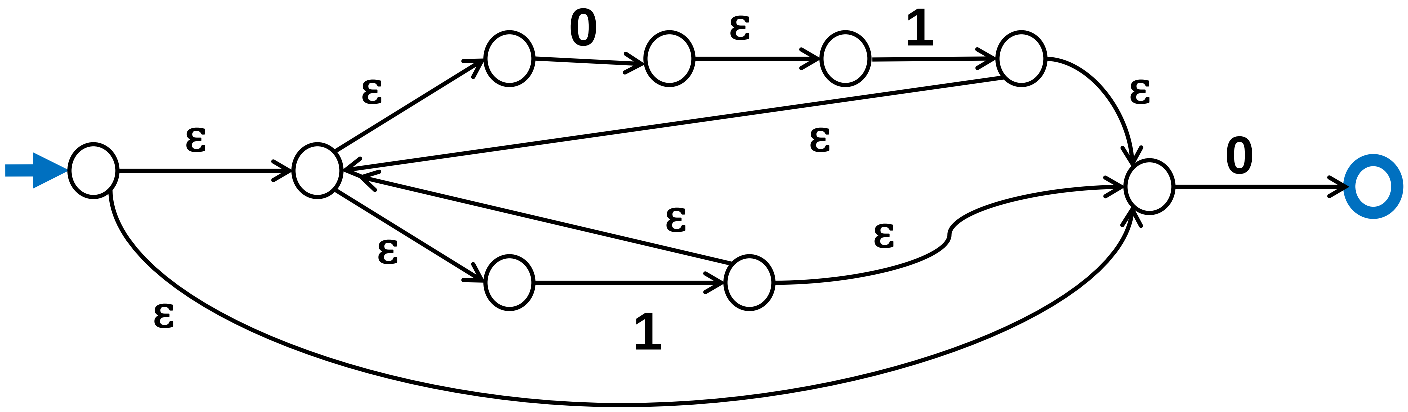
Proof idea: Structural induction based on the recursive definition of regular expressions...



# build an NFA for $(01 \cup 1)^*0$



**(01 ∪1)*0**



RE → NFA

# NFAs vs. DFAs

Every DFA **is** an NFA

– DFAs have requirements that NFAs don't have

Can NFAs recognize more languages?

# NFAs vs. DFAs

Every DFA **is** an NFA

– DFAs have requirements that NFAs don't have

Can NFAs recognize more languages? No!

**Theorem:** For every NFA there is a DFA that recognizes exactly the same language.

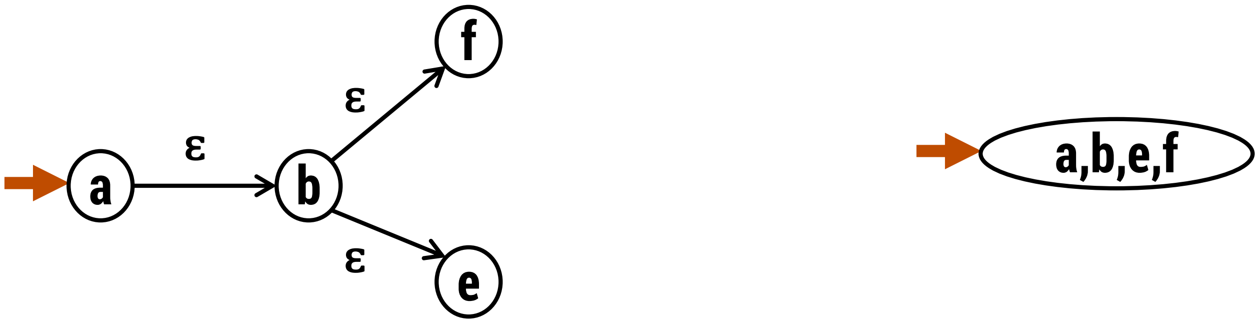
# conversion of NFAs to DFAs

Proof Idea:

- The DFA keeps track of ALL the states that the part of the input string read so far can reach in the NFA
- There will be one state in the DFA for each *subset* of states of the NFA that can be reached by some string

## New start state for DFA

– The set of all states reachable from the start state of the NFA using only edges labeled **ε**



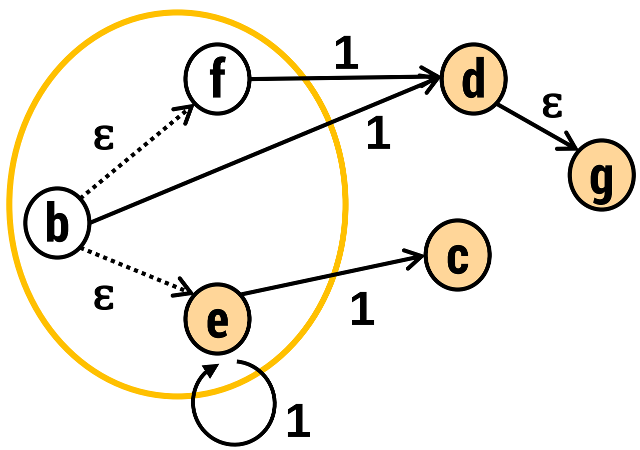

NFA

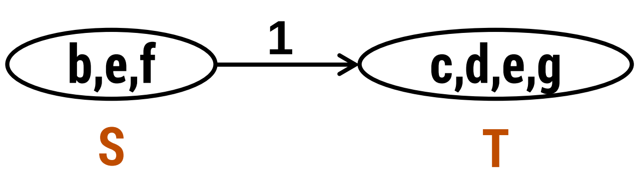
DFA

# conversion of NFAs to a DFAs

For each state of the DFA corresponding to a set S of states of the NFA and each symbol **s**

- Add an edge labeled **s** to state corresponding to T, the set of states of the NFA reached by
  starting from some state in S, then
  following one edge labeled by **s**, and
  then following some number of edges labeled by ε
- T will be ∅ if no edges from S labeled **s** exist



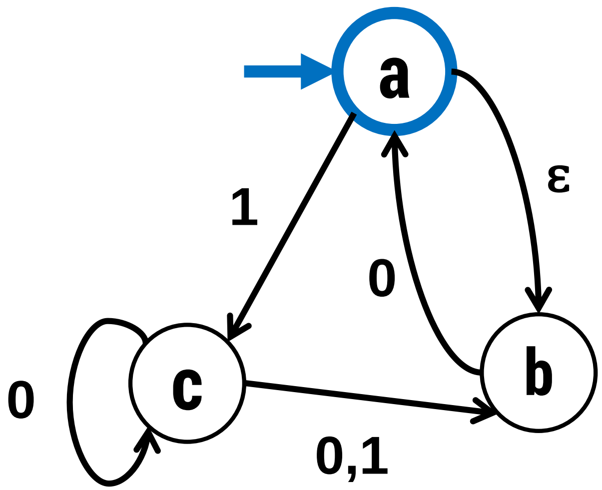
## Final states for the DFA

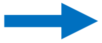
- All states whose set contain some final state of the NFA


NFA

DFA

# example: NFA to DFA

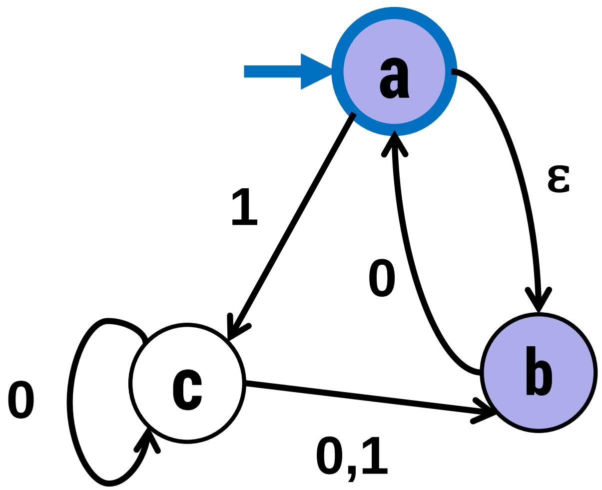


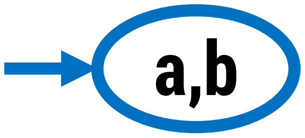
NFA

DFA

# example: NFA to DFA

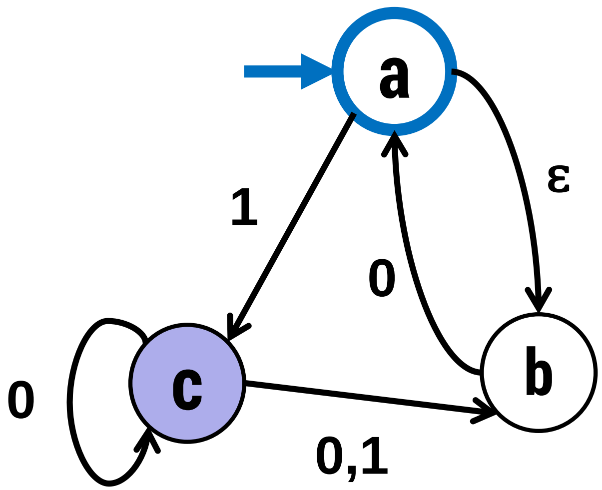



NFA

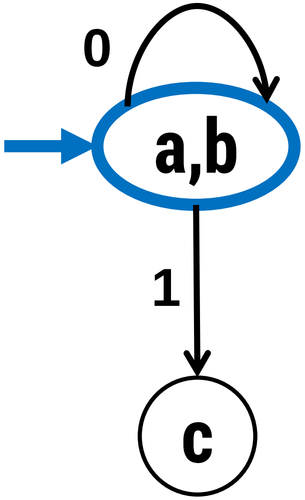



DFA

# example: NFA to DFA

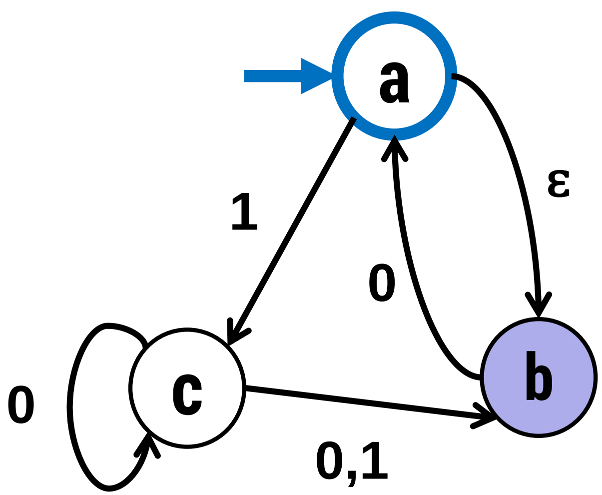



NFA

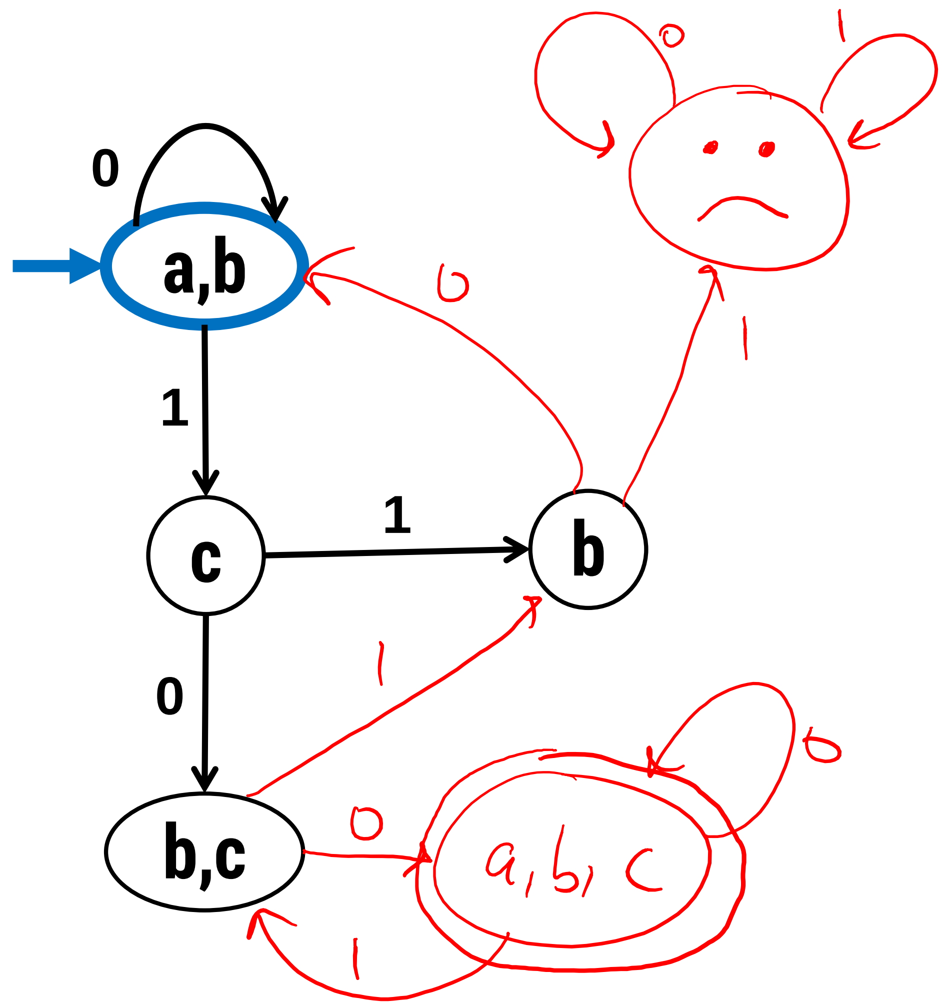



DFA

# example: NFA to DFA

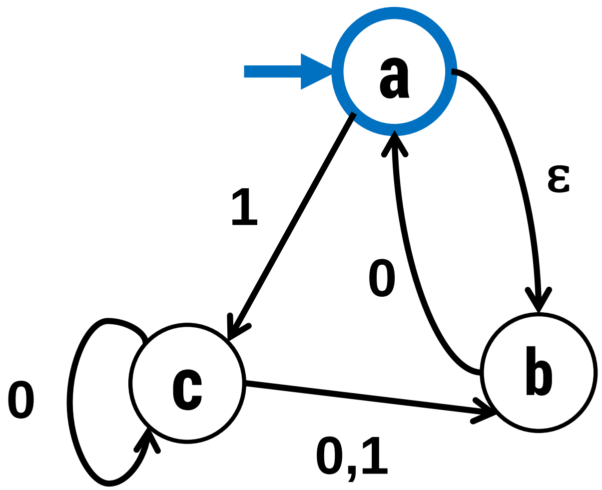



NFA

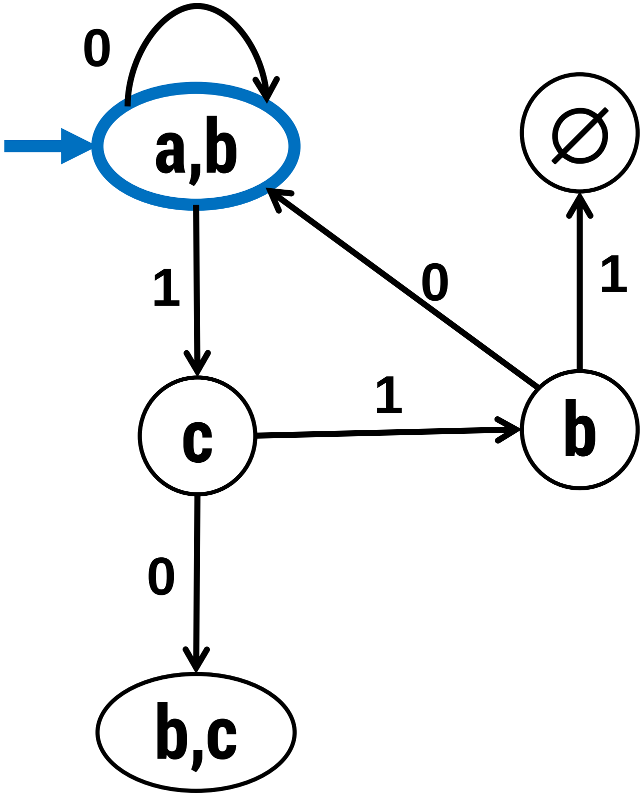



DFA

## example: NFA to DFA

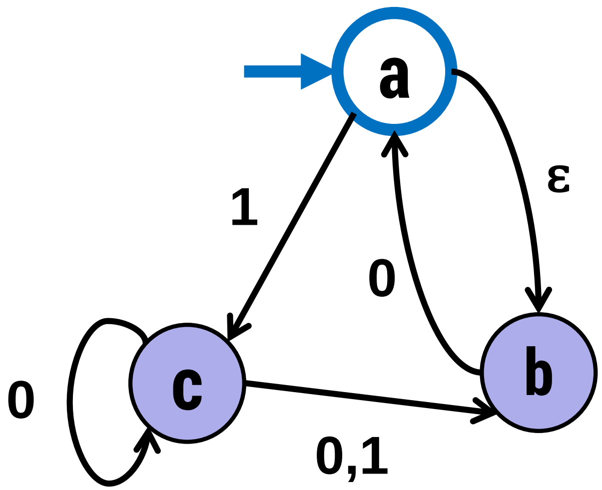



NFA

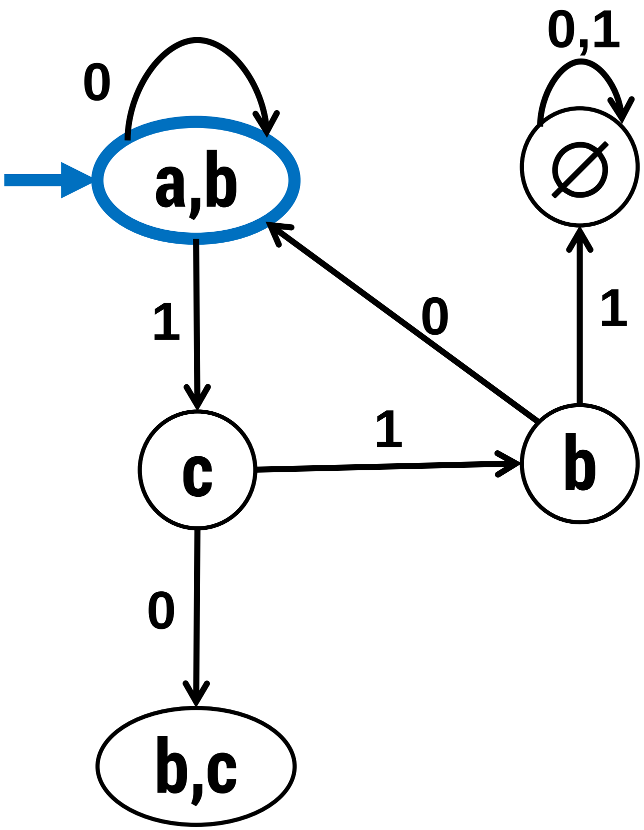



DFA

# example: NFA to DFA

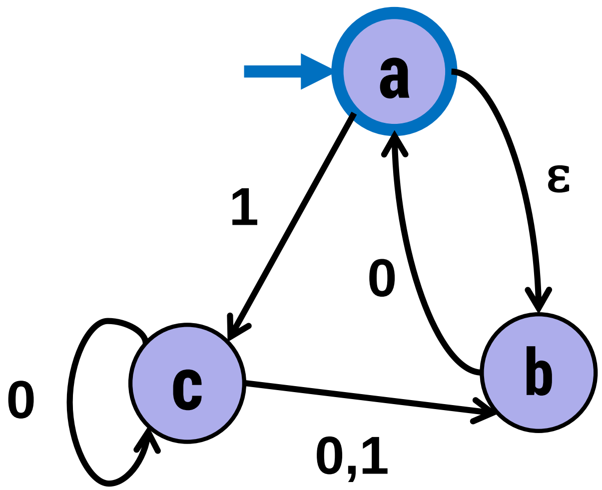



NFA

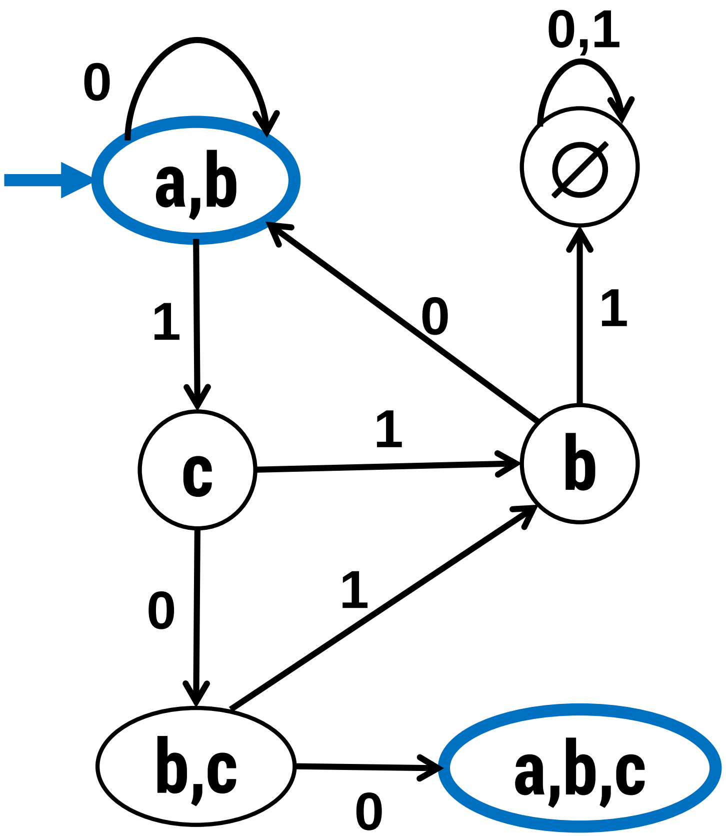



DFA

# example: NFA to DFA

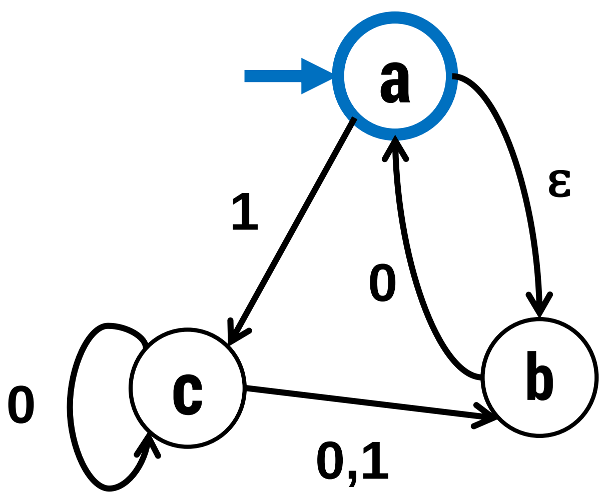



NFA

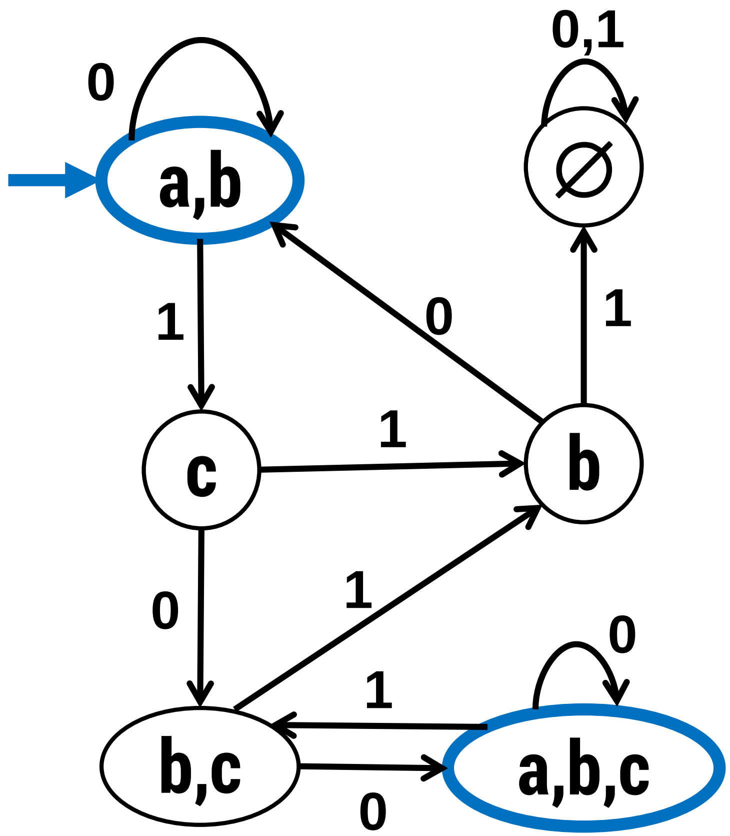



DFA

# example: NFA to DFA




NFA




DFA

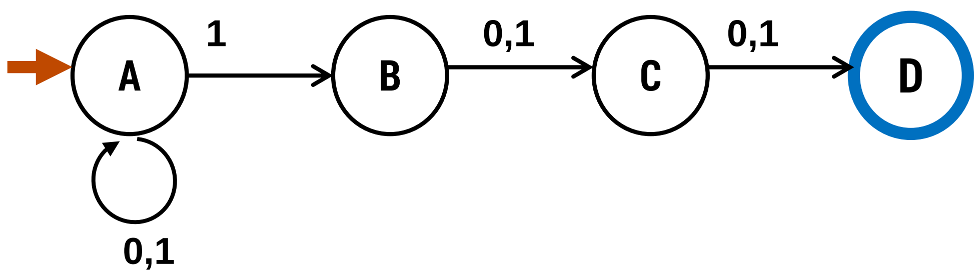
# exponential blow-up in simulating mondeterminism

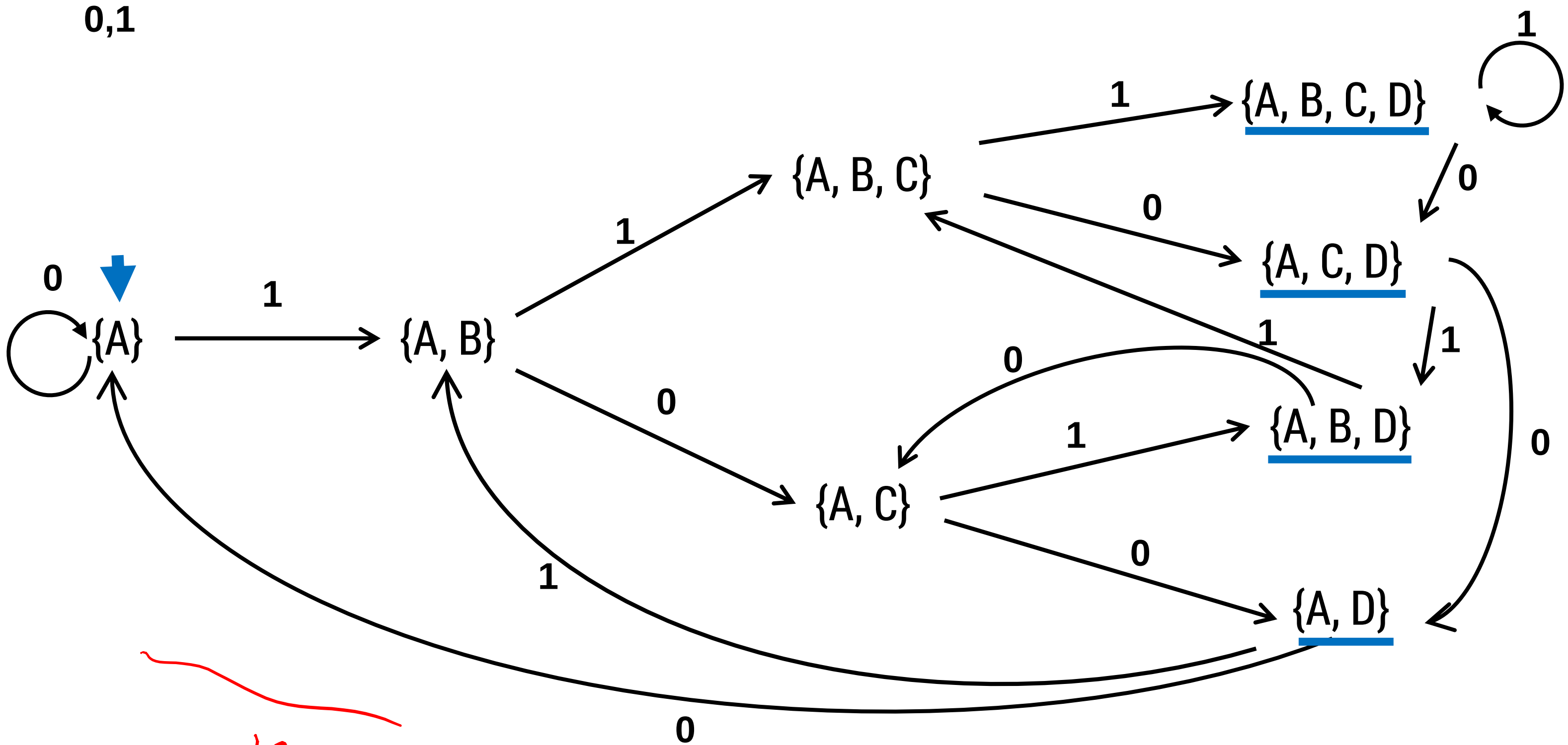
- In general the DFA might need a state for every subset of states of the NFA
  - Power set of the set of states of the NFA
  - n-state NFA yields DFA with at most $2^n$ states
  - We saw an example where roughly $2^n$ is necessary

    Is the $n^{th}$ char from the end a 1?

- The famous "P=NP?" question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms

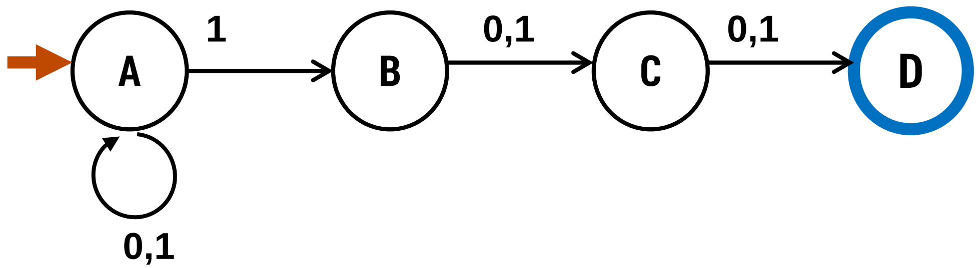
# 1 in ~~third~~ nth position from end

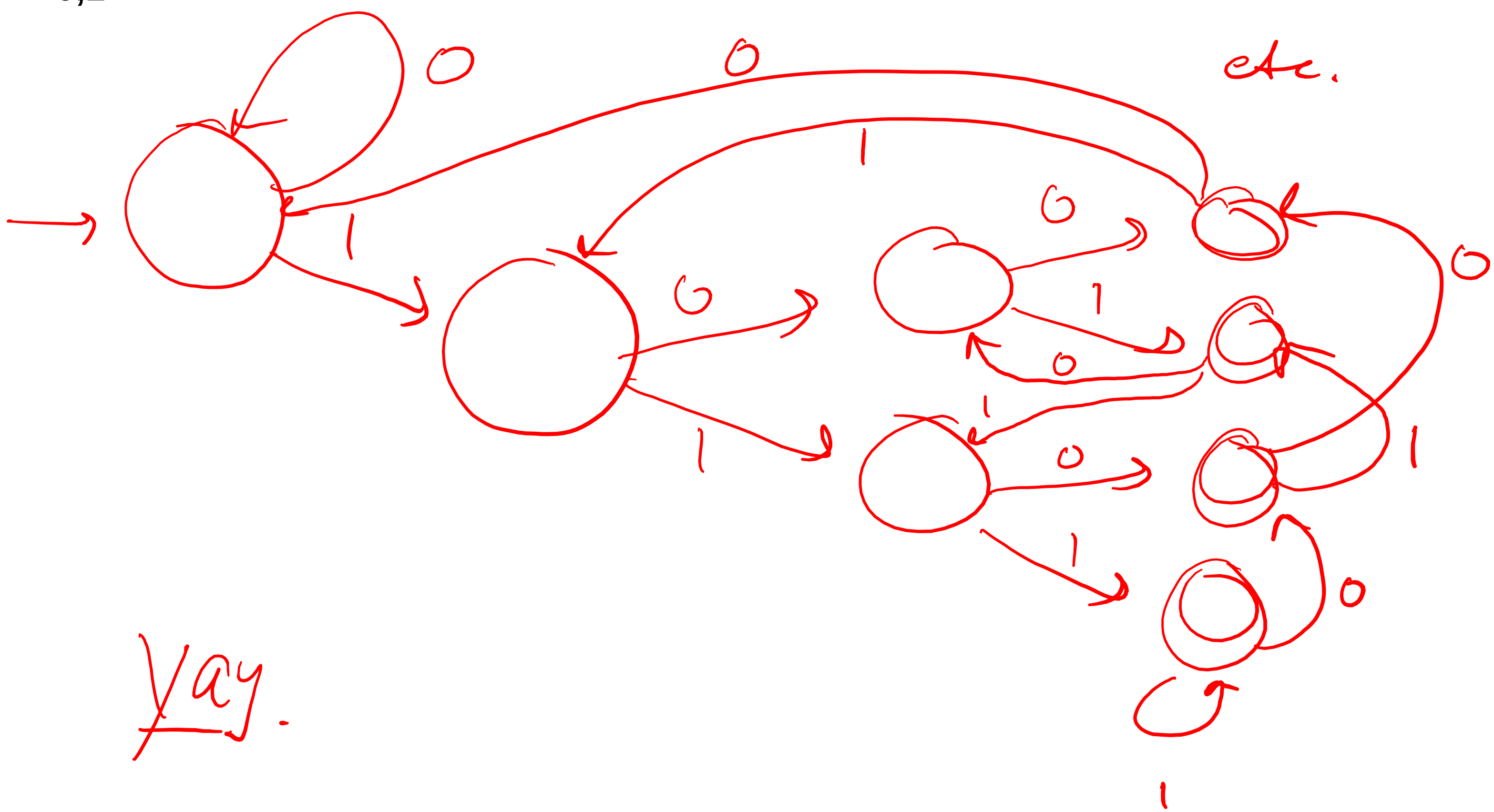


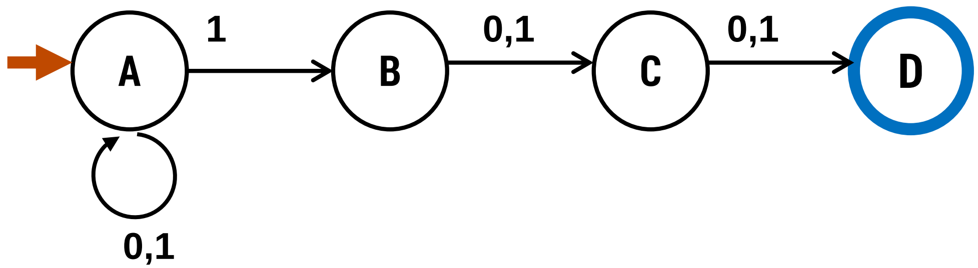

$2^n$ states

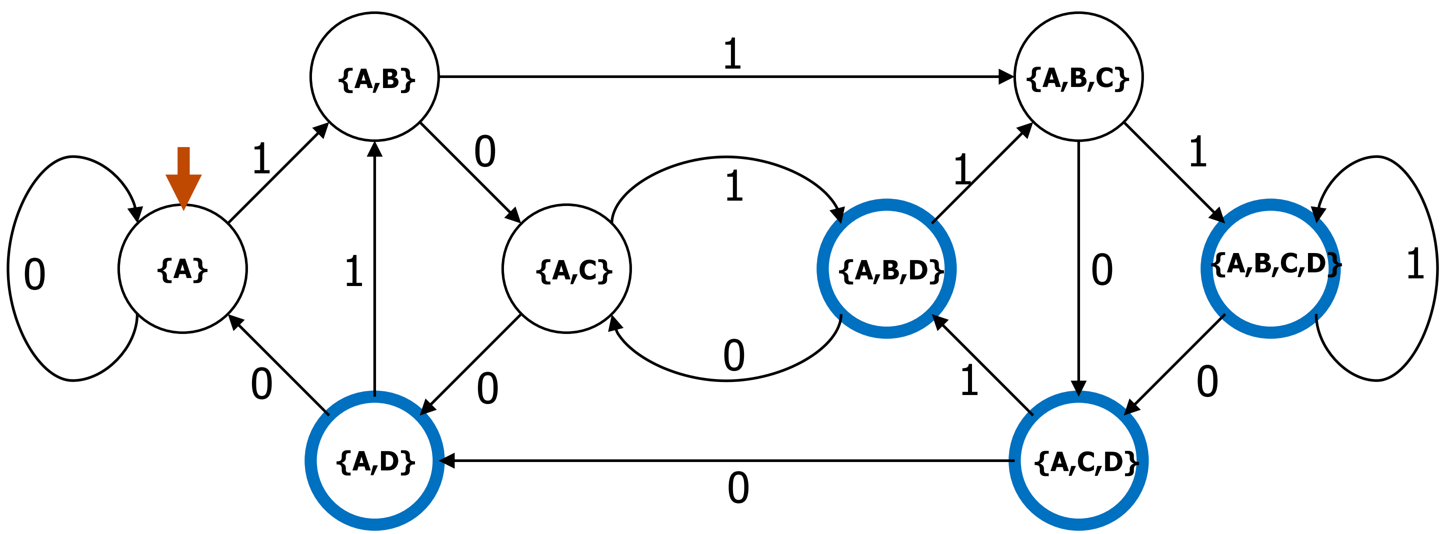
# 1 in third position from end



# 1 in third position from end



# DFAs ≡ regular expressions

We have shown how to build an optimal DFA for every regular expression

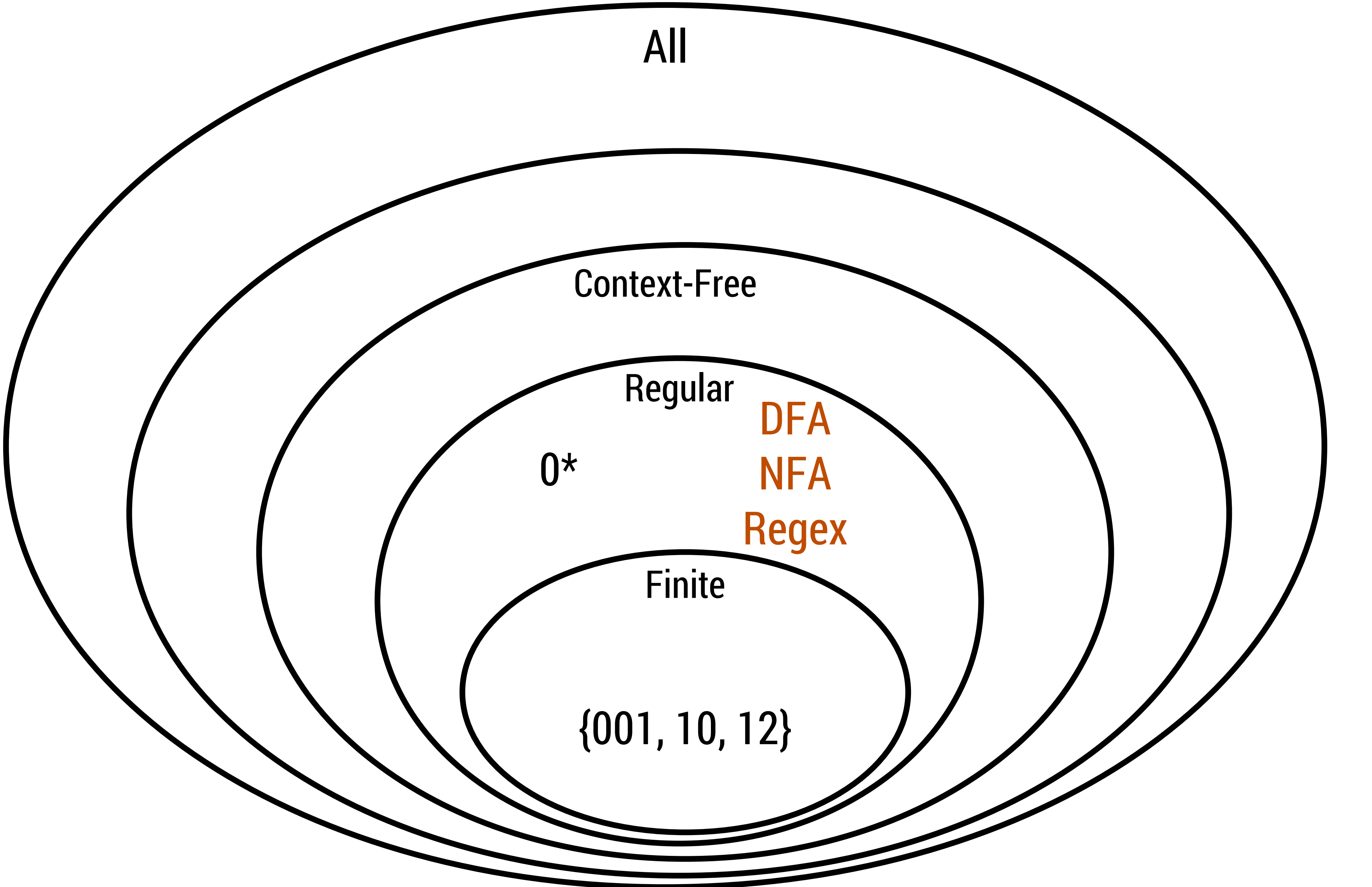
RE → NFA → DFA

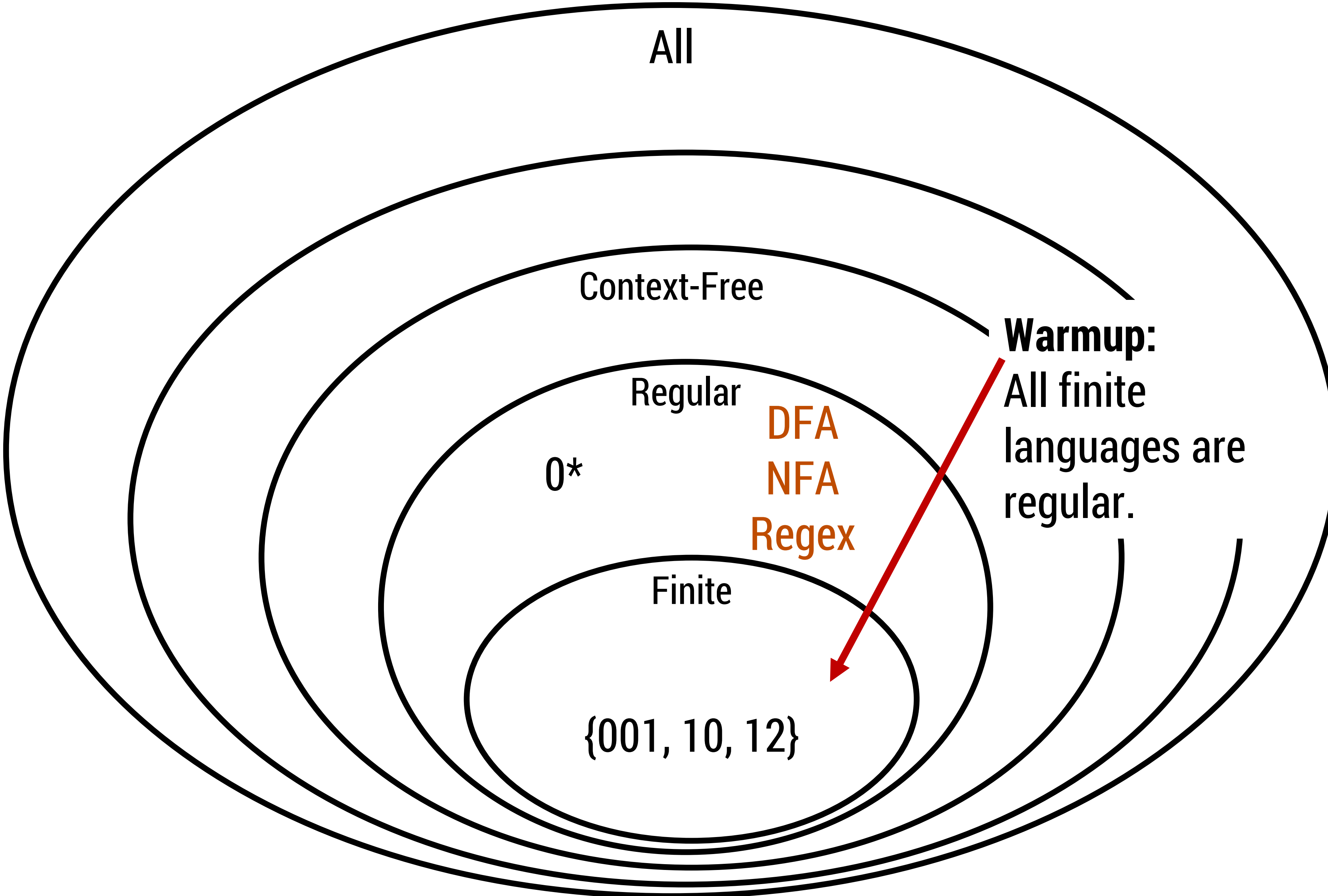
- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Theorem: A language is recognized by a DFA if and only if it has a regular expression.

We show the other direction of the proof at the end of these lecture slides.

# languages and machines!




All
Context-Free
Regular
0*
DFA
NFA
Regex
Finite
{001, 10, 12}
Warmup:
All finite languages are regular.

# DFAs recognize any finite language

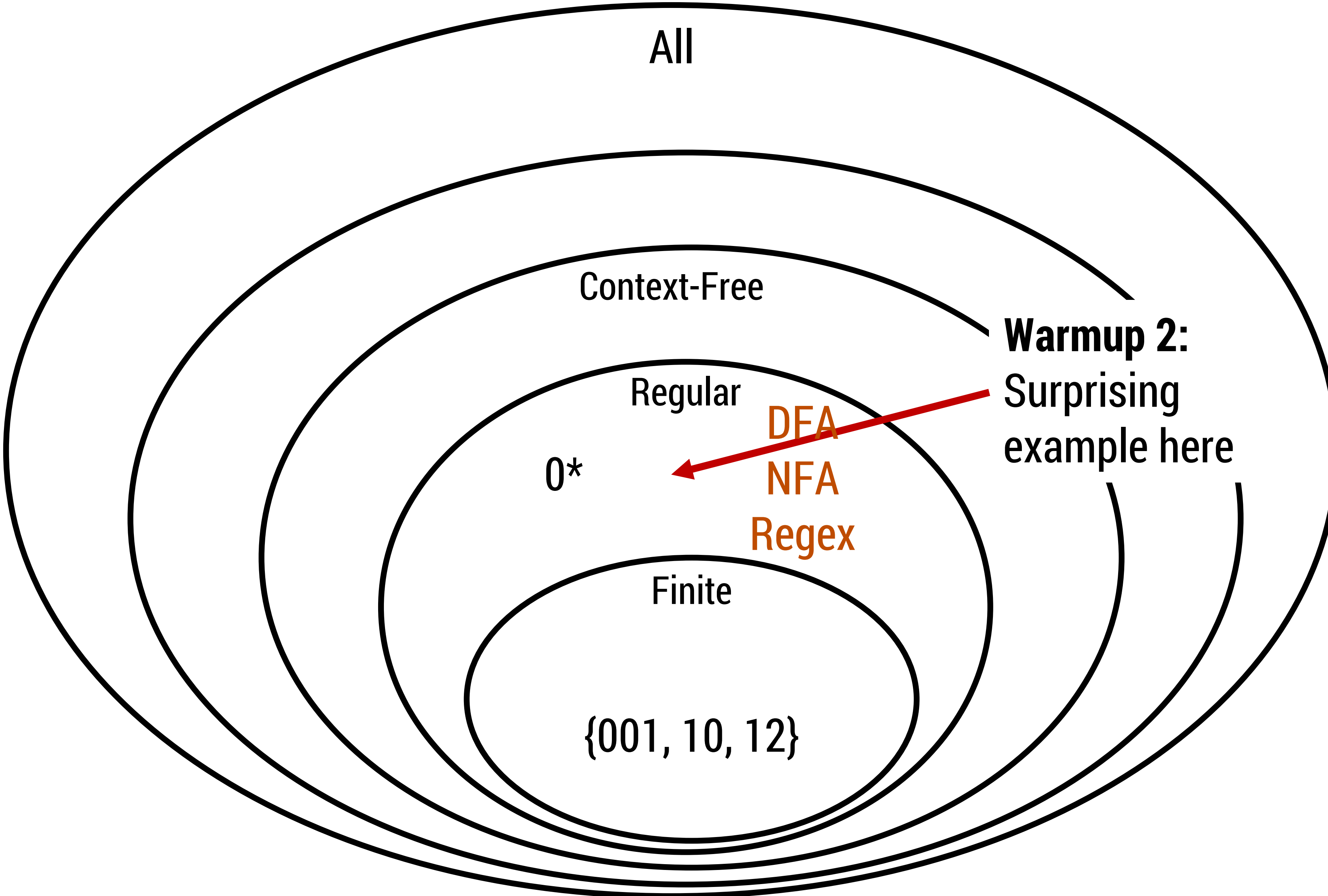
~~Exercise: Hard code it into the DFA.~~

$L \subseteq \Sigma^*$

$L = \{w_1, w_2, \ldots, w_k\}$

$w_1 \mid w_2 \mid \cdots \mid w_k$

# languages and machines!



$L = \{x \in \{0,1\}^*: x$ has an equal number of substrings 01 and 10$\}$.

L is infinite. $\{0, 00, 000, 0000, \ldots$

$010, 01010, 000001010, \ldots\}$

L is regular.

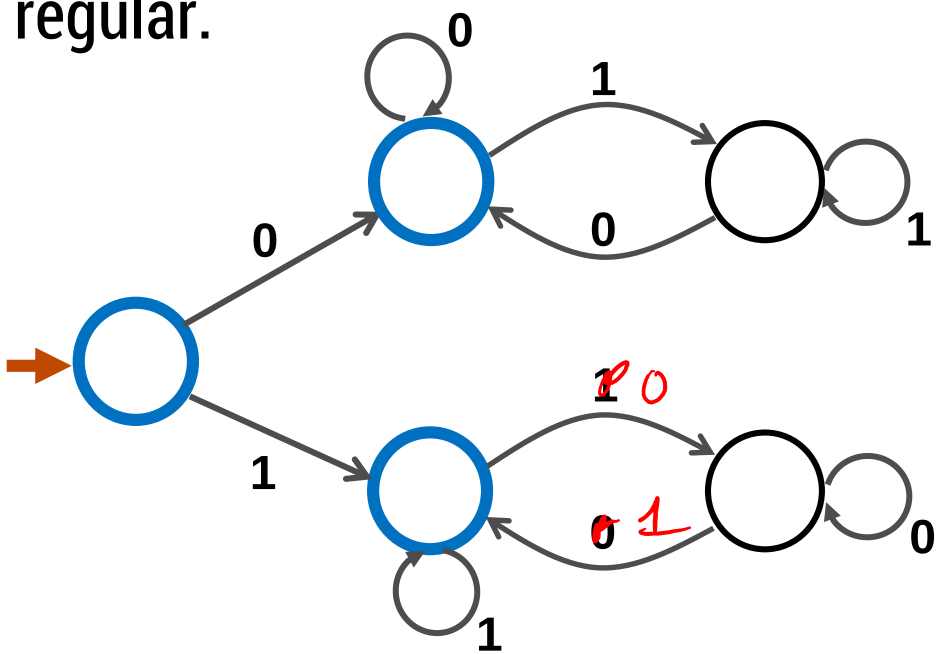
$((01)(1^* \cup 0^*)10)^* \mid (10(1^* \cup 0^*)01)^*$

# an interesting **regular** language

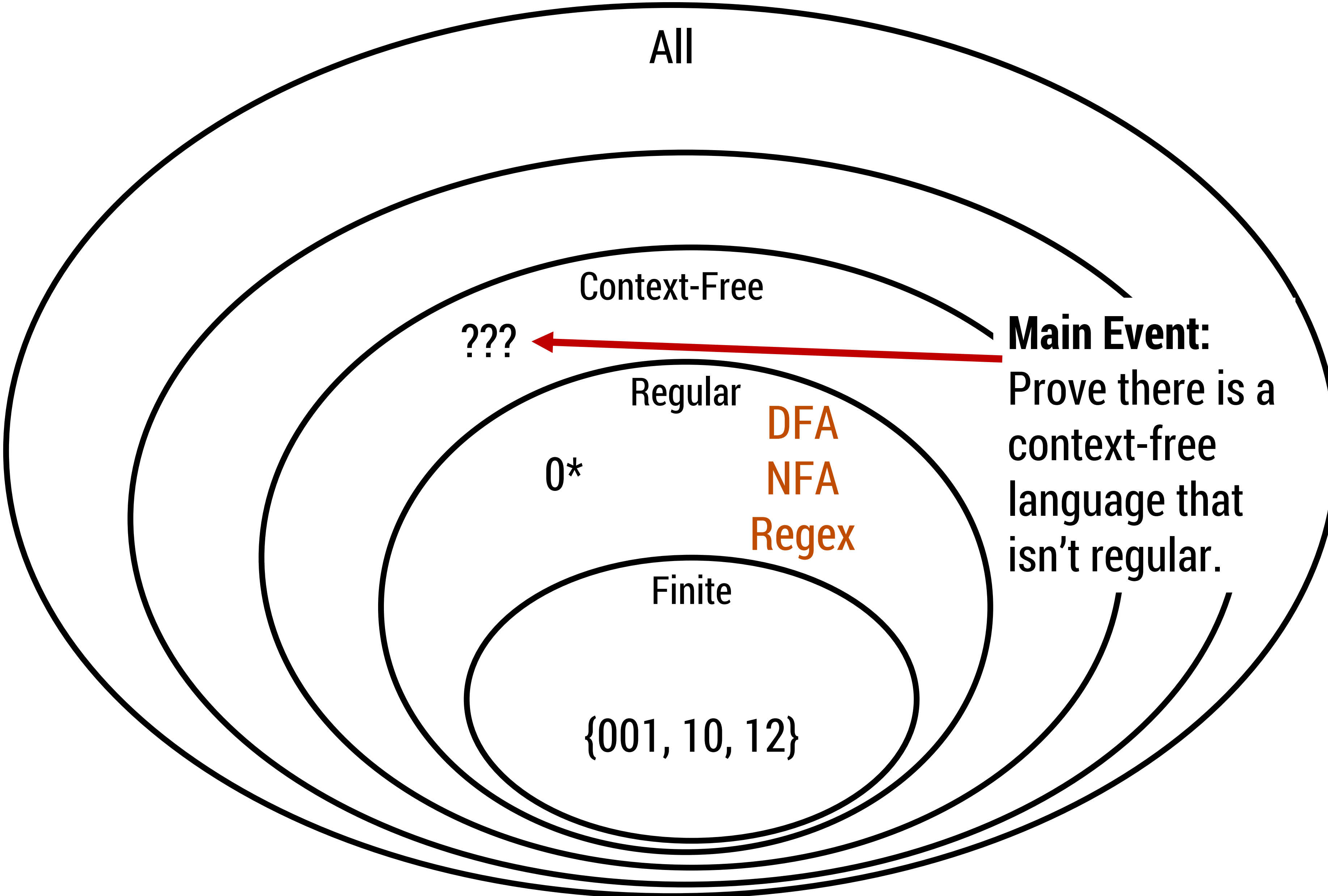
$L = \{x \in \{0,1\}^*: x \text{ has an equal number of substrings } 01 \text{ and } 10\}.$

L is infinite.

L is regular.



# languages and machines!



# DFAs ≡ regular expressions

Theorem: A language is recognized by a DFA if and only if it has a regular expression

Proof: Last class: RegExp $\to$ NFA $\to$ DFA

Now: NFA $\to$ RegExp

(Enough to show this since every DFA is also an NFA.)

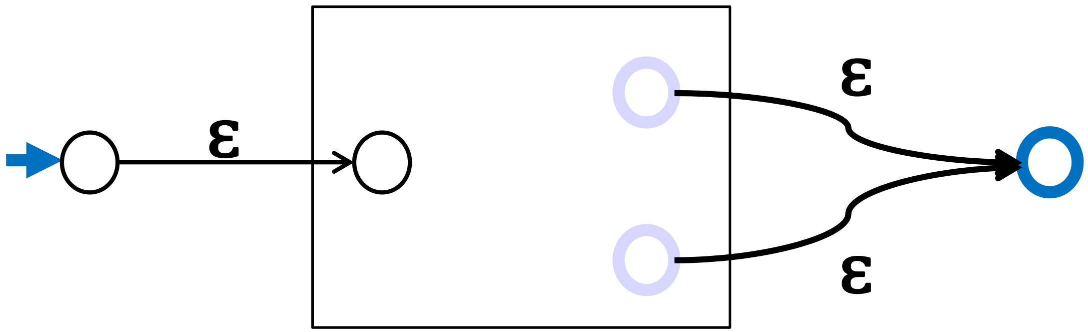
# generalized NFAs

- Like NFAs but allow
  - Parallel edges
  - Regular Expressions as edge labels

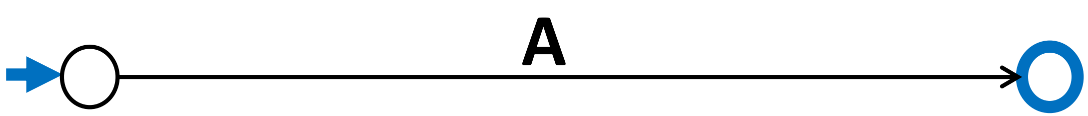
    NFAs already have edges labeled **ε** or ***a***

- An edge labeled by **A** can be followed by reading a string of input chars that is in the language represented by **A**
- A string x is accepted iff there is a path from start to final state labeled by a regular expression whose language contains x

Add new start state and final state



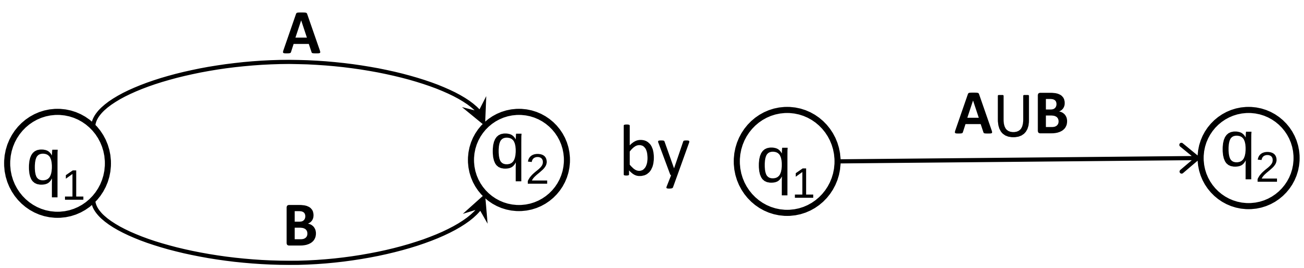

Then eliminate original states one by one, keeping the same language, until it looks like:



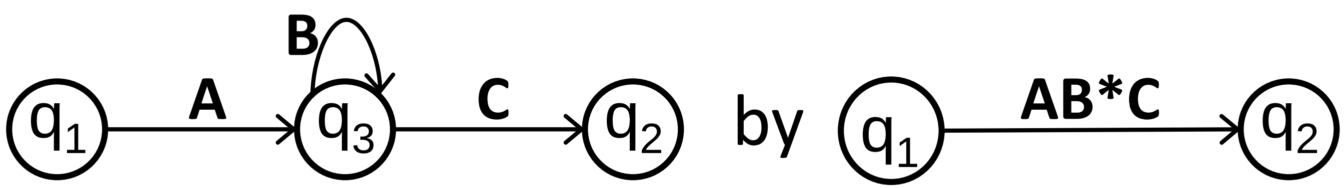

Final regular expression will be **A**

# only two simplification rules

- Rule 1: For any two states $q_1$ and $q_2$ with parallel edges (possibly $q_1$=$q_2$), replace




- Rule 2: Eliminate non-start/final state $q_3$ by replacing all

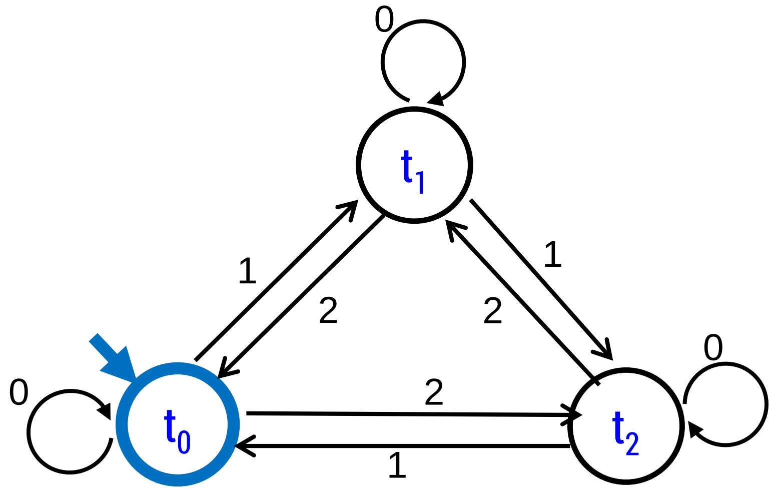



for *every* pair of states $q_1$, $q_2$ (even if $q_1$=$q_2$)

# converting an NFA to a regular expression

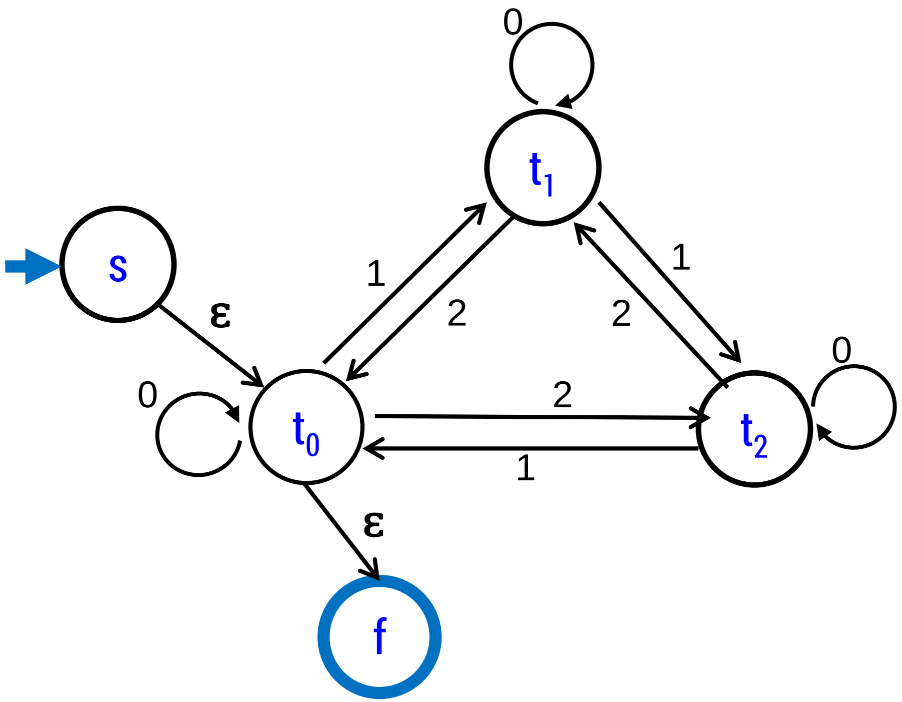
Consider the DFA for the mod 3 sum

– Accept strings from {0,1,2}* where the digits mod 3 sum of the digits is 0



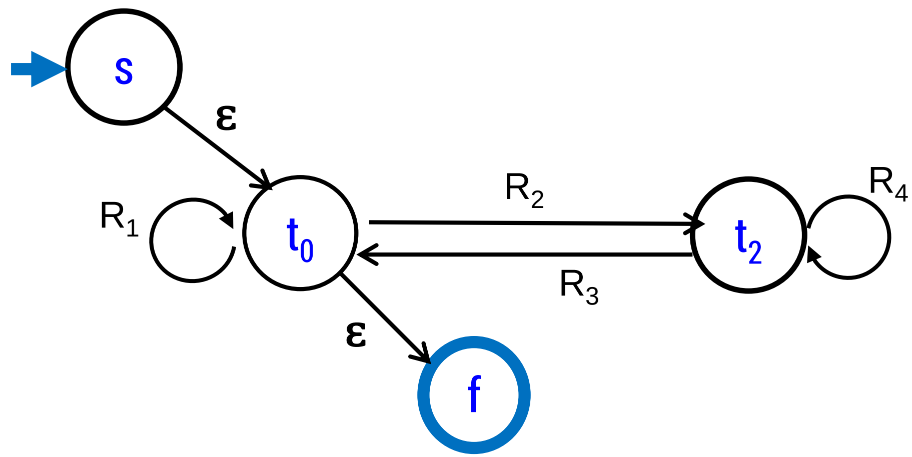
## Label edges with regular expressions

$t_0 \rightarrow t_1 \rightarrow t_0$ : 10*2
$t_0 \rightarrow t_1 \rightarrow t_2$ : 10*1
$t_2 \rightarrow t_1 \rightarrow t_0$ : 20*2
$t_2 \rightarrow t_1 \rightarrow t_2$ : 20*1

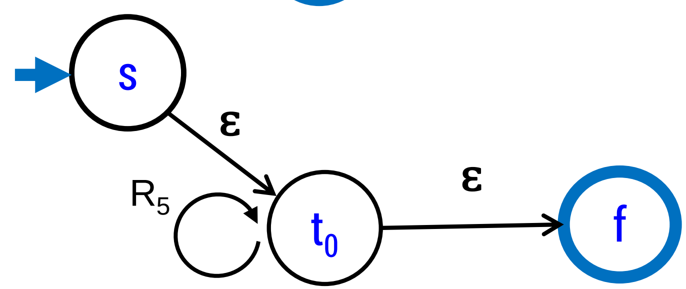


# finite automaton without $t_1$

$R_1$: $0 \cup 10^*2$
$R_2$: $2 \cup 10^*1$
$R_3$: $1 \cup 20^*2$
$R_4$: $0 \cup 20^*1$




$R_5$: $R_1 \cup R_2R_4{}^*R_3$




Final regular expression:
$(0 \cup 10^*2 \cup (2 \cup 10^*1)(0 \cup 20^*1)^*(1 \cup 20^*2))^*$