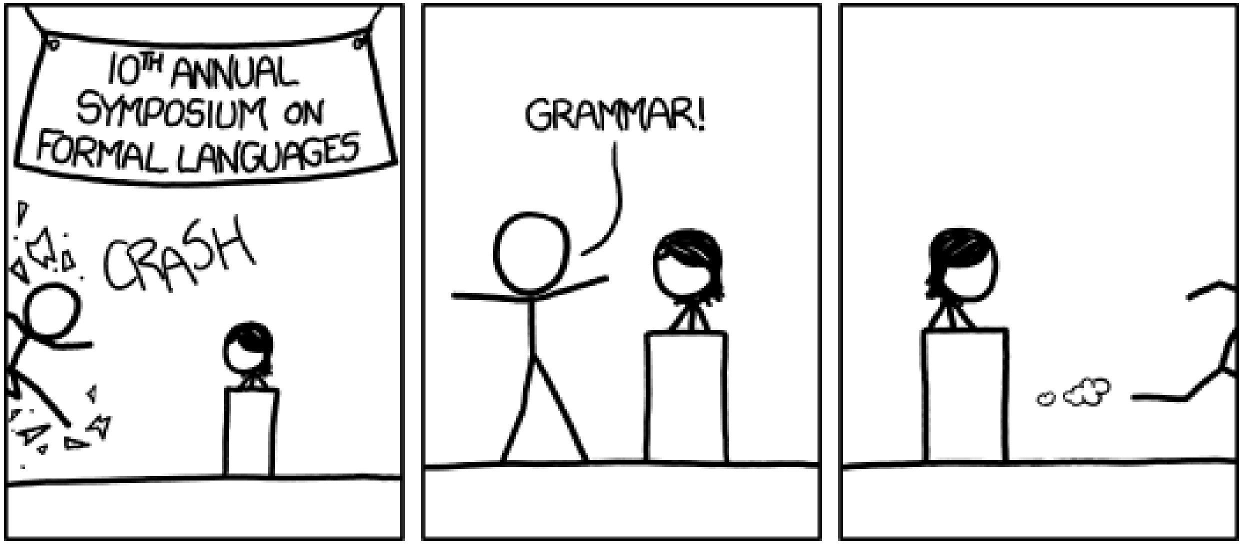
Spring 2015

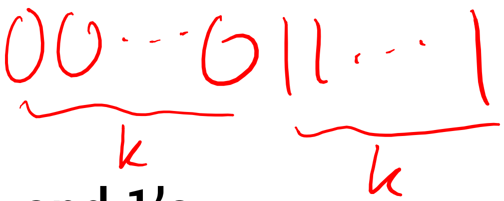
Lecture 21: Context-free grammars and finite state machines



limitations of regular expressions

- Not all languages can be specified by regular expressions

- Even some easy things like
  - Palindromes
  - Strings with equal number of 0's and 1's



- But also more complicated structures in programming languages
  - Matched parentheses
  - Properly formed arithmetic expressions
  - etc.

- A Context-Free Grammar (CFG) is given by a finite set of substitution rules involving
  - A finite set **V** of *variables* that can be replaced
  - Alphabet Σ of *terminal symbols* that can't be replaced
  - One variable, usually **S**, is called the *start symbol*

- The rules involving a variable **A** are written as

$$\mathbf{A} \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$$

where each w_i is a string of variables and terminals:

$$w_i \in (\mathbf{V} \cup \Sigma)^*$$

how CFGs generate strings

- Begin with start symbol **S**

- If there is some variable **A** in the current string you can replace it by one of the w's in the rules for **A**
  - $\mathbf{A} \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$ | w
  - Write this as **xAy ⇒ xwy**
  - Repeat until no variables left

- The set of strings the CFG generates are all strings produced in this way that have no variables

Example: $S \to 0S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$

S → ε
S → 0
S → 1

S → 0S0
→ 01S10
→ 010S010
→ 010010

Example: $S \to 0S \mid S1 \mid \varepsilon$

G

S → 0S → 0S1 → 01

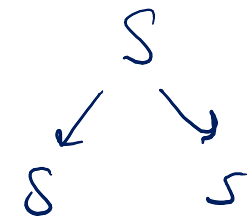
$L_G = \{0^*1^*\}$

Grammar for $\{0^n 1^n : n \geq 0\}$

(all strings with same # of 0's and 1's with all 0's before 1's)

$S \rightarrow 0S1$

$S \rightarrow \varepsilon$



Example: $\mathbf{S \rightarrow (S) \mid SS \mid \varepsilon}$

$\Sigma = \{ (,) \}$

$()(()())$.

$S \rightarrow SS$

$\rightarrow (S)S \rightarrow ()S \rightarrow ()(S)$

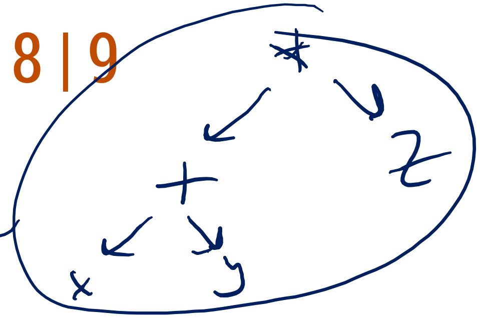
$\rightarrow ()(SS) \rightarrow ()((S)S)$

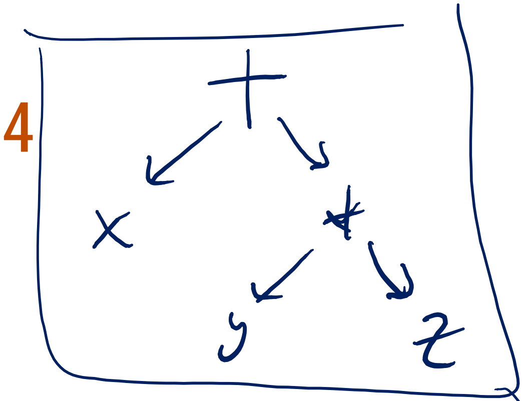
$\rightarrow ()((S)(S))$

$\rightarrow ()(()(S)) \rightarrow ()(()())$.

simple arithmetic expressions

E → E+E | E∗E | (E) | x | y | z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9





Generate (2∗x) + y

E → E+E → E + y
→ (E) + y
→ (E*E) + y
→ (2*E)+y
→ (2*x)+y

Generate x+y∗z in two fundamentally different ways

E → E+E → x+E → x+E*E → x+y*E → x+y*z

E → E*E → E*z → E+E*z → x+E*z
→ x+y*z.

parse trees

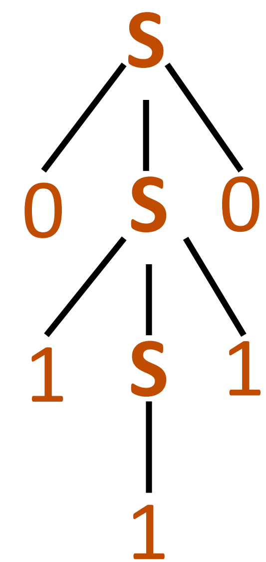
Suppose that grammar G generates a string x

A **parse tree** of x for G has

- Root labeled S (start symbol of G)
- The children of any node labeled **A** are labeled by symbols of w left-to-right for some rule **A** $\rightarrow$ w
- The symbols of x label the leaves ordered left-to-right

$$\mathbf{S} \rightarrow 0\mathbf{S}0 \mid 1\mathbf{S}1 \mid 0 \mid 1 \mid \varepsilon$$

Parse tree of 01110:



CFGs and recursively-defined sets of strings

- A CFG with the start symbol **S** as its only variable recursively defines the set of strings of terminals that **S** can generate

- A CFG with more than one variable is a simultaneous recursive definition of the sets of strings generated by *each* of its variables
  - Sometimes necessary to use more than one

building precedence in simple arithmetic expressions

E → T → F*T   (x+y)*z

- **E** – expression (start symbol)
- **T** – term **F** – factor **I** – identifier **N** - number

E → **T** | **E+T**

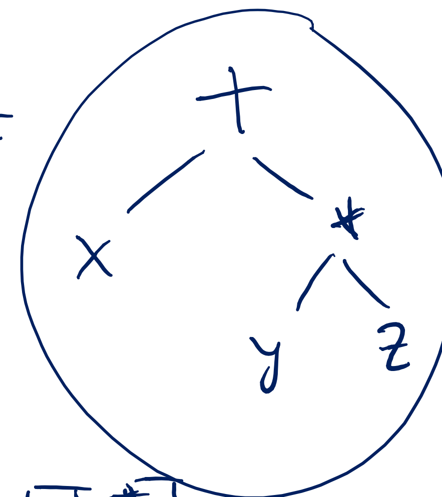
T → **F** | **F∗T**

F → (**E**) | **I** | **N**

I → x | y | z

N → 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

x + y * z



E → E + T
→ T + T
→ F + T
→ I + T → x+T → x + F*T → x + F*F → x + I*F → x + I*I → x + y*I → x + y*z

Backus-Naur form (same as CFG)

BNF (Backus-Naur Form) grammars

- Originally used to define programming languages
- Variables denoted by long names in angle brackets, e.g.

  <identifier>, <if-then-else-statement>,

  <assignment-statement>, <condition>

  $::=$ used instead of $\rightarrow$

BNF for C

```
statement:
  ((identifier | "case" constant-expression | "default") ":")*
  (expression? ";" |
   block |
   "if" "(" expression ")" statement |
   "if" "(" expression ")" statement "else" statement |
   "switch" "(" expression ")" statement |
   "while" "(" expression ")" statement |
   "do" statement "while" "(" expression ")" ";" |
   "for" "(" expression? ";" expression? ";" expression? ")" statement |
   "goto" identifier ";" |
   "continue" ";" |
   "break" ";" |
   "return" expression? ";"
  )

block: "{" declaration* statement* "}"

expression:
  assignment-expression%

assignment-expression: (
    unary-expression (
      "=" | "*=" | "/=" | "%=" | "+=" | "-=" | "<<=" | ">>=" | "&=" |
      "^=" | "|="
    )
  )* conditional-expression

conditional-expression:
  logical-OR-expression ( "?" expression ":" conditional-expression )?
```

parse trees

Back to middle school:

<sentence>::=<noun phrase><verb phrase>

<noun phrase>::==<article><adjective><noun>

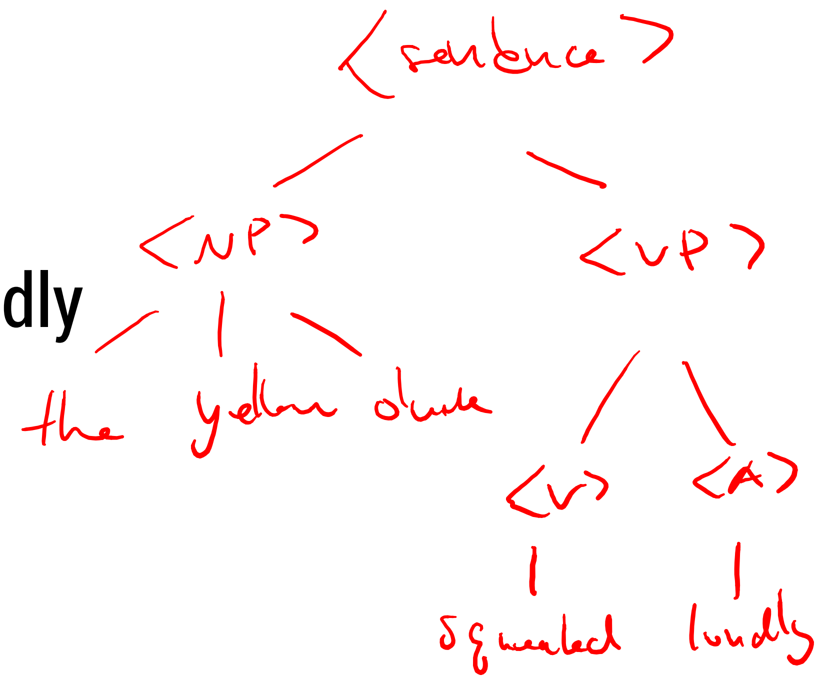
<verb phrase>::=<verb><adverb>|<verb><object>

<object>::=<noun phrase>

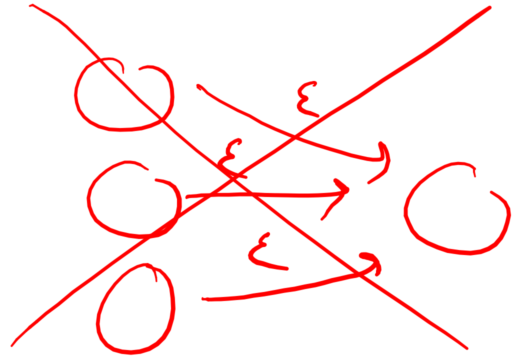
Parse:

The yellow duck squeaked loudly

The red truck hit a parked car



- States
- Transitions on inputs
- Start state and final states



- The language recognized by a machine is the set of strings that reach a final state

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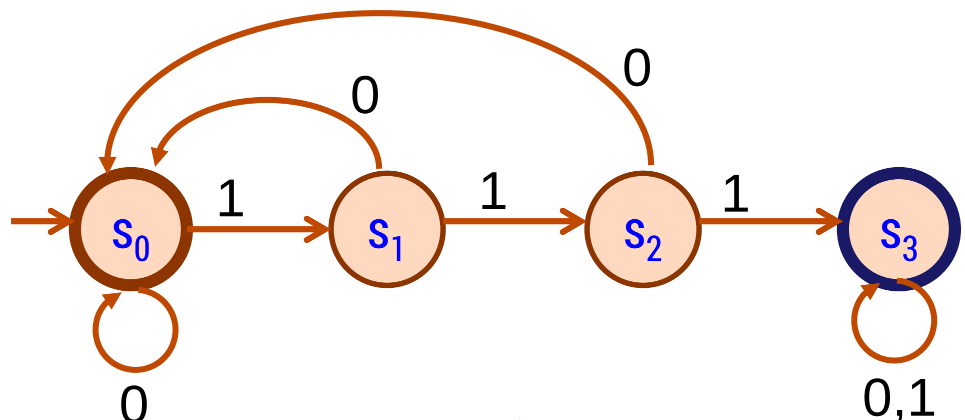
State	0	1
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_3
s_3	s_3	s_3



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applications of FSMs (aka finite automata)

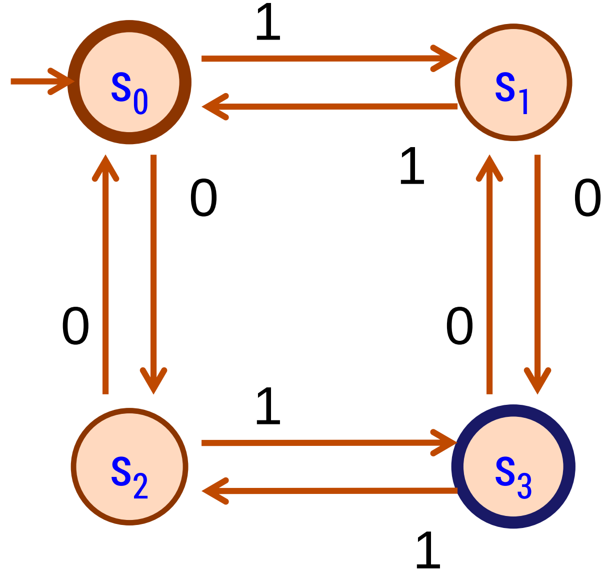
- Implementation of regular expression matching in programs like `grep`
- Control structures for sequential logic in digital circuits
- Algorithms for communication and cache-coherence protocols
  - Each agent runs its own FSM
- Design specifications for reactive systems
  - Components are communicating FSMs

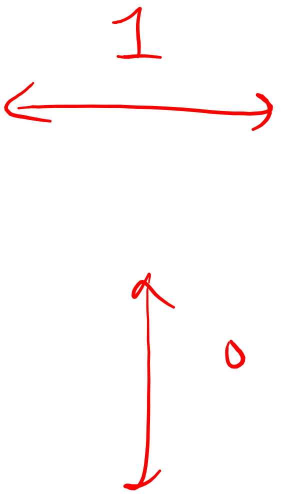
applications of FSMs (aka finite automata)

- Formal verification of systems
  - Is an unsafe state reachable?
- Computer games
  - FSMs provide worlds to explore
- Minimization algorithms for FSMs can be extended to more general models used in
  - Text prediction
  - Speech recognition

what language does this machine recognize?

odd #0's and
odd #1's





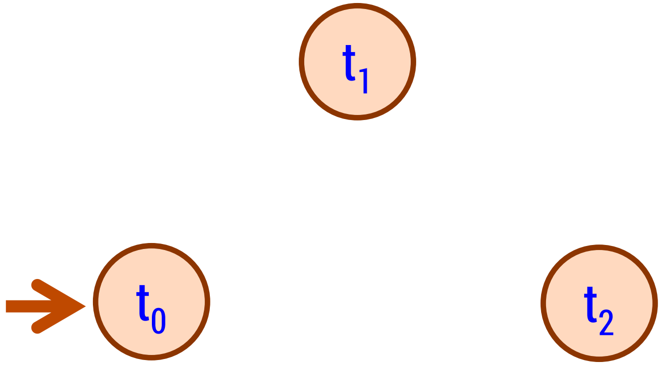
can we recognize these languages with DFAs?

- $\emptyset$
- Σ^*
- $\{ x \in \{0,1\}^* : \text{len}(x) > 1\}$

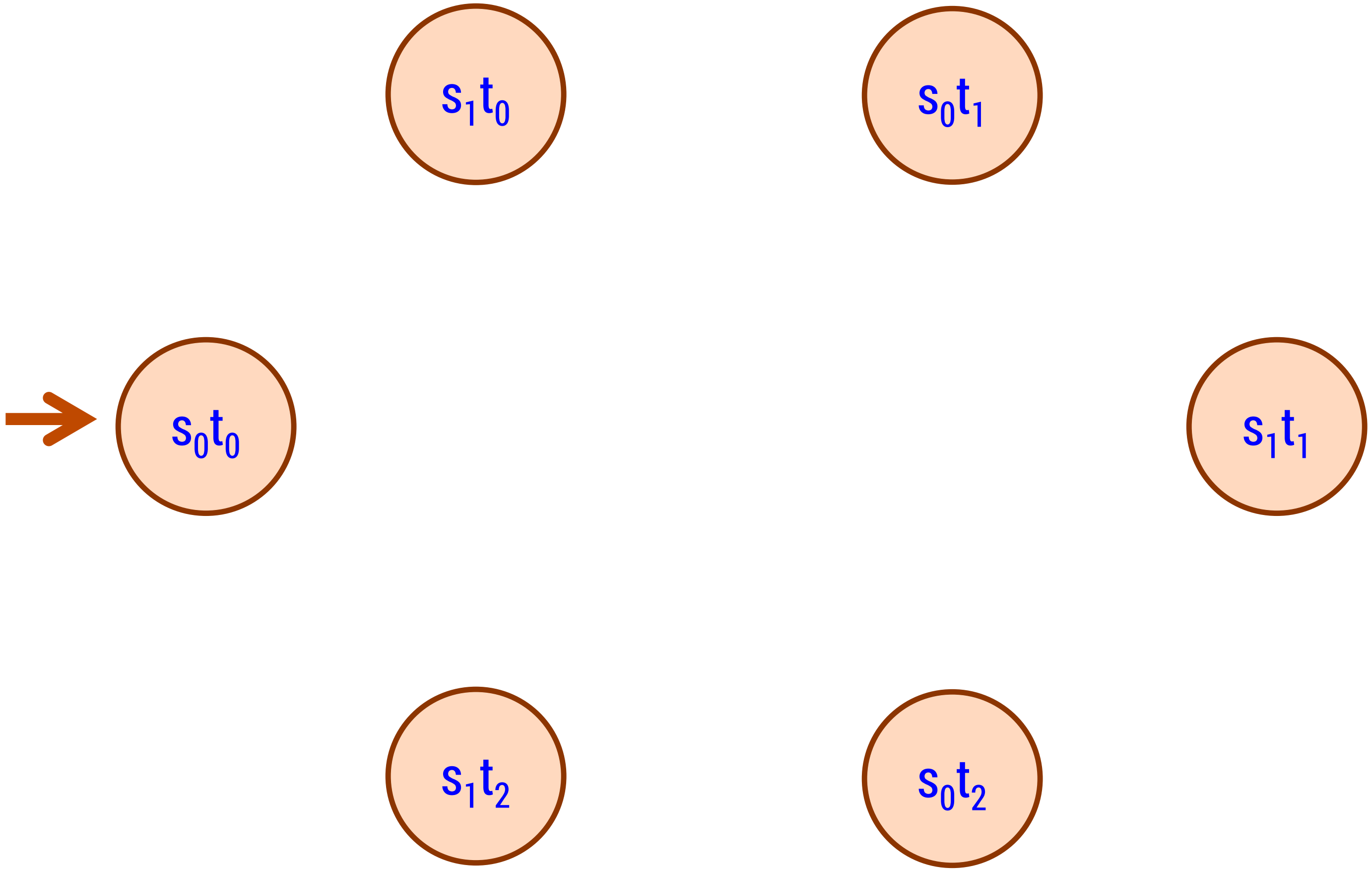
FSM that accepts binary strings with a 1 three positions from the end

M_1: Strings with an even number of 2's

M_2: Strings where the sum of digits mod 3 is 0



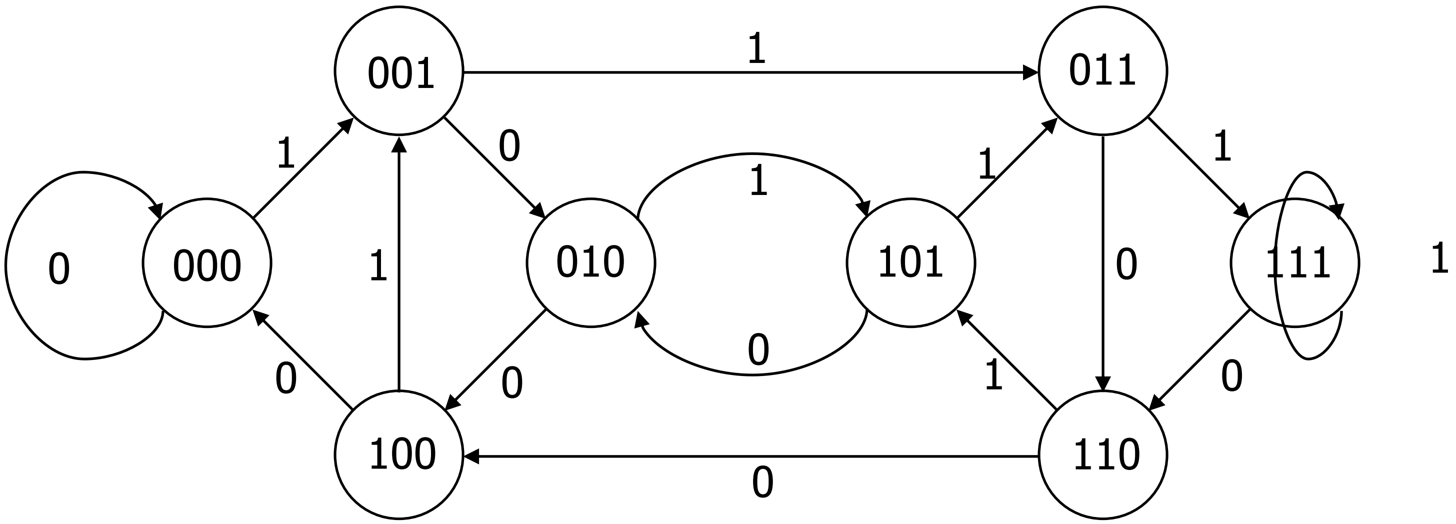
both: even number of 2's and sum mod 3 = 0

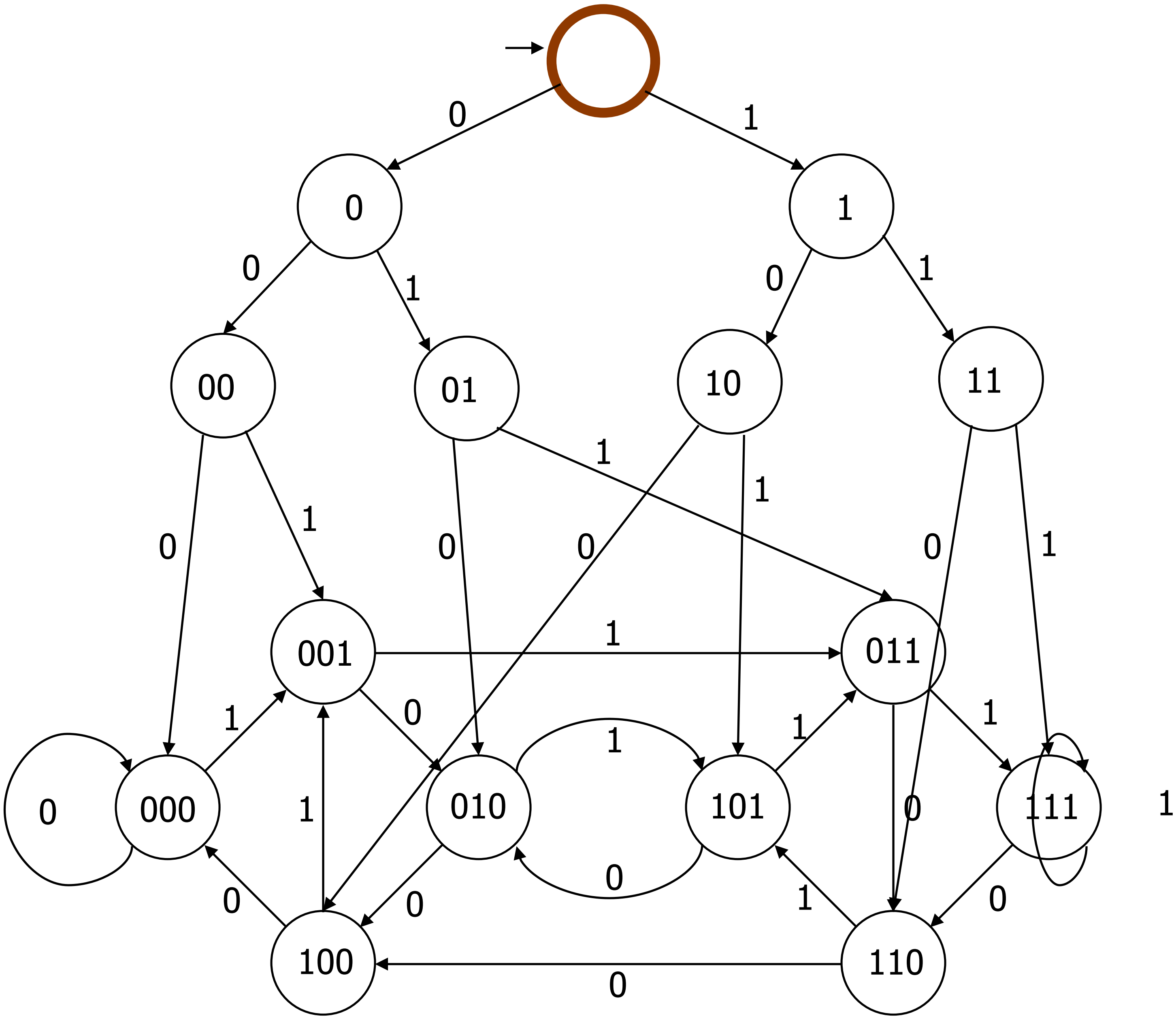


DFA that accepts strings of a’s, b’s, c’s with no more than 3 a’s

“Remember the last three bits”

3 bit shift register





0
1
0
1
0
1
0
1
00
01
10
11
001
011
000
010
101
111
100
110