cse 311: foundations of computing

Spring 2015
Lecture 18:
Recursively defined sets and structural induction

administrative

Four weeks left: What happens now?

The class speeds up a bit.
Homework problems get more conceptual.

We will cover:

- Recursively defined sets and functions
- Structural induction
- Regular expressions and context free grammars
- Relations and graphs
- Finite state machines and automata
- Turing machines and undecidability

recursive definition of sets

Recursive definition

- **Basis step:** $0 \in S$
- **Recursive step:** if $x \in S$, then $x + 2 \in S$
- **Exclusion rule:** Every element in S follows from basis steps and a finite number of recursive steps

recursive definition of sets

Basis: $6 \in S$; $15 \in S$;
Recursive: if $x, y \in S$, then $x + y \in S$;

Basis: $[1, 1, 0] \in S, [0, 1, 1] \in S$;
Recursive:
if $[x, y, z] \in S,\ \alpha \in \mathbb{R}$, then $[\alpha x, \alpha y, \alpha z] \in S$
if $[x_1, y_1, z_1], [x_2, y_2, z_2] \in S$
then $[x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2] \in S$

Powers of 3:

recursive definitions of sets: general form

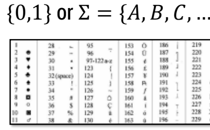
Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

strings

- An *alphabet* Σ is any finite set of characters.

  e.g. $\Sigma = \{0,1\}$ or $\Sigma = \{A, B, C, \ldots X, Y, Z\}$ or

  $\Sigma =$ 

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\mathcal{E} \in \Sigma^*$ ($\mathcal{E}$ is the empty string)
  - **Recursive:** if $w \in \Sigma^*, a \in \Sigma$, then $wa \in \Sigma^*$

palindromes

Palindromes are strings that are the same backwards and forwards.

Basis:
ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:
If p is a palindrome then apa is a palindrome for every $a \in \Sigma$.

binary strings such that…

First digit cannot be a 1.

* No occurrence of the substring 11.

function definitions on recursively defined sets

Length:
len (ε) = 0;
len (wa) = 1 + len(w); for $w \in \Sigma^*, a \in \Sigma$

Reversal:
$\varepsilon^{\mathrm{R}} = \varepsilon$
$(wa)^{\mathrm{R}} = aw^{\mathrm{R}}$ for $w \in \Sigma^*, a \in \Sigma$

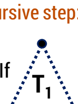
Concatenation:
$x \bullet \varepsilon = x$ for $x \in \Sigma^*$
$x \bullet wa = (x \bullet w)a$ for $x, w \in \Sigma^*, a \in \Sigma$

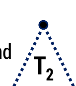
function definitions on recursively defined sets

Number of vowels in a string:
$\Sigma = \{a, b, c, \ldots, z\}$
$\mathcal{V} = \{a, e, i, o, u\}$

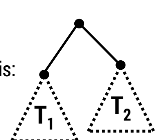
rooted binary trees

- Basis: • is a rooted binary tree
- Recursive step:

If 

and 

are rooted binary trees,

then so is: 

rooted binary trees

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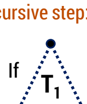
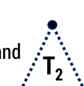
If 

and 

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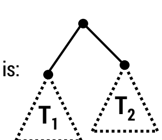
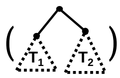
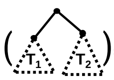
then so is: 

defining a function on rooted binary trees

- size(•) = 1
- size  = 1 + size(T_1) + size(T_2)
- height(•) = 0
- height  =1 + max{height(T_1), height(T_2)}

structural induction

How to prove $\forall\ x \in S,\ P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\ x \in S,\ P(x)$

structural induction

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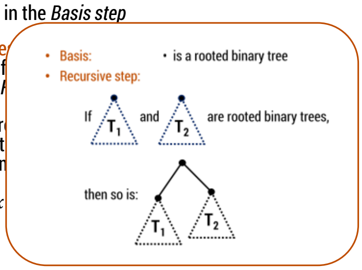
Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothe
arbitrary values of
mentioned in the *R*

Inductive Step: Pr
elements w const
named elements n

Conclude that $\forall\ x$



structural induction vs. ordinary induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k+1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Let $Q(n)$ be true iff for all $x \in S$ that take n recursive steps to be constructed, $P(x)$ is true.

using structural induction

Let S be given by:

- **Basis:** $6 \in S$; $15 \in S$;
- **Recursive:** if $x, y \in S$ then $x + y \in S$.

Claim: Every element of S is divisible by 3.