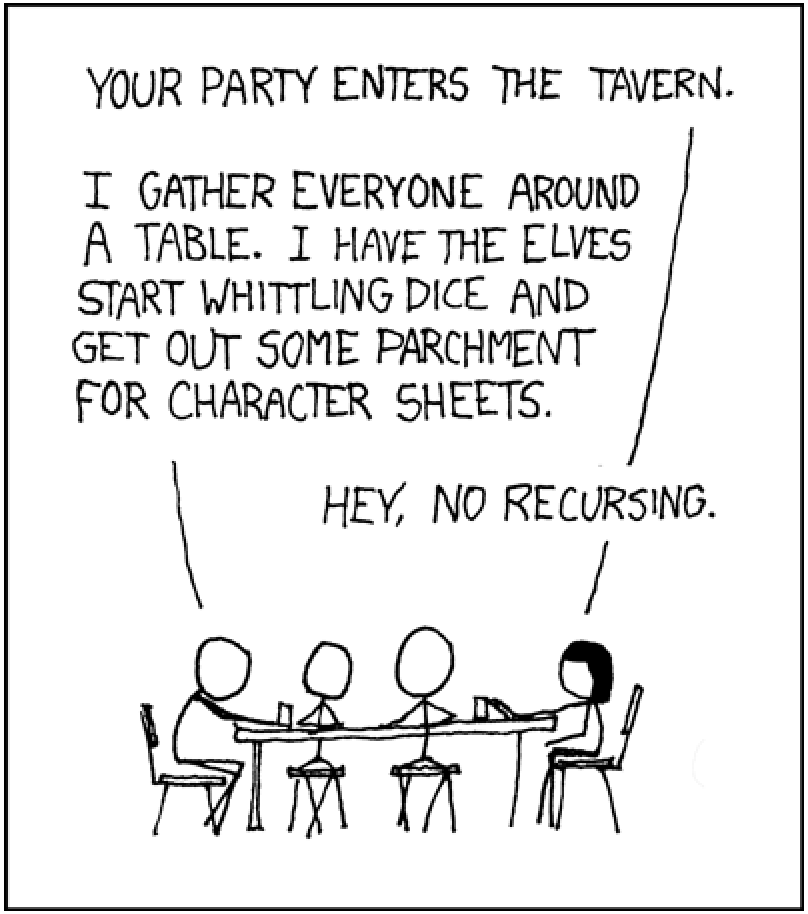
Spring 2015

Lecture 17: Recursively defined sets



administrative

Midterm review session tonight @ 6pm (EEB 105)

MIDTERM FRIDAY (IN THIS ROOM, USUAL TIME)

- Closed book.
- One page (front and back) of hand-written notes allowed.
- Exam includes induction and strong induction!
- Homework #5 is up now, but due on Friday, May 15th.

review: strong induction

$$P(0)$$

$$\forall k \; \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \to P(k+1)\Big)$$

$$\therefore \forall n \, P(n)$$

Follows from ordinary induction applied to

$$Q(n) = P(0) \land P(1) \land P(2) \land \cdots \land P(n)$$

review: strong induction English proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

$k+1 = ab$

We argue by strong induction.

P(n) = "n can be expressed as a product of primes" for $n \geq 2$.

Base Case:

Note that 2 is prime; so, we can express it as "2" which is a product of primes.

Induction Hypothesis:

Suppose $P(2) \wedge P(3) \wedge \cdot \cdot \cdot \wedge P(k)$ is true for some $k \geq 2$.

Induction Step:

We go by cases.

Suppose k+1 is prime. Then, "k+1" is a product of primes.

Suppose k+1 is composite. Then, $k+1 = ab$ for some a and b such that $1 < a, b < k+1$.

By our IH, we know $a = p_1 p_2 \cdots p_m$ and $b = q_1 q_2 \cdots q_n$.

So, $k+1 = ab =$ "$p_1 p_2 \cdots p_m q_1 q_2 \cdots q_n$", which is a product of primes.

Thus, our claim is true for $n \geq 2$ by strong induction.

review: recursive definition of functions

- $F(0) = 0$; $F(n+1) = F(n) + 1$ **for all** $n \geq 0$

  $F(n) = n$

- $G(0) = 1$; $G(n+1) = 2 \times G(n)$ **for all** $n \geq 0$

  $G(n) = 2^n$

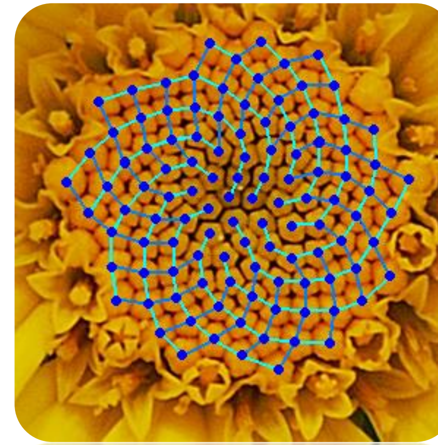
- $0! = 1$; $(n+1)! = (n+1) \times n!$ **for all** $n \geq 0$

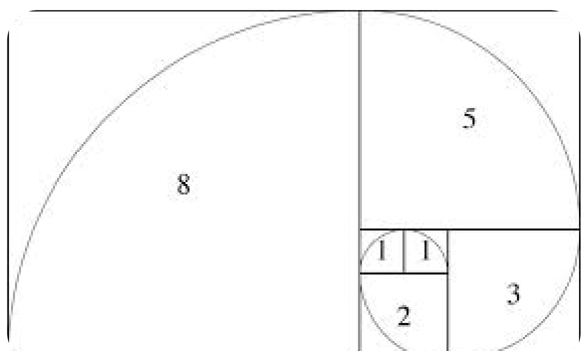
- $H(0) = 1$; $H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$

  $H(n) = \left.2^{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}\right\} n$

review: Fibonacci numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \quad \textbf{for all } n \geq 2$$





Theorem: $f_n < 2^n$ **for all** $n \geq 2$.

bounding the Fibonacci numbers

Theorem: $\boxed{2^{\frac{n}{2}-1} \leq f_n < 2^n}$ for all $n \geq 2$

$f_3 = f_2 + f_1$

$= 2 < 2^3$

$2^{0.5} \leq$

$P(n) = $ " $2^{\frac{n}{2}-1} \leq f_n < 2^n$ "

<u>Base case:</u> $P(2)$: $f_2 = f_1 + f_0 = 1$ | $\underbrace{2^{\frac{2}{2}-1}}_{1} \leq 1 < 2^2$ ✓

$P(3)$:

<u>IH:</u> Assume $P(j)$ for all $2 \leq j \leq k$ for some $k \geq 3$

<u>IS:</u> ~~$f_{k+1} = f_k + f_{k-1}$~~

$\overset{IH}{<} \underbrace{2^k}_{j=k} + \underbrace{2^{k-1}}_{j=k-1}$

$< 2^k + 2^k = 2 \cdot 2^k$

$= 2^{k+1}$.

| By induction $\forall n \geq 2$ $P(n)$.

$f_{k+1} = f_k + f_{k-1}$

$\overset{IH}{\geq} 2^{\frac{k}{2}-1} + 2^{\frac{k-1}{2}-1} \overset{?}{\geq} 2^{\frac{k+1}{2}-1}$

$\geq 2^{-1}\left(2^{\frac{k}{2}} + 2^{\frac{k-1}{2}}\right)$

$= 2^{\frac{k+1}{2}-1}\left(2^{-1/2} + 2^{-1}\right) \geq 2^{\frac{k+1}{2}-1}$ □

b/c $\frac{1}{\sqrt{2}} + \frac{1}{2} \geq \frac{1}{2} + \frac{1}{2} = 1$

running time of Euclid's algorithm

$a > b$

$$\gcd(a,b) = \gcd(b, a \bmod b)$$

$$\gcd(a,0) = a.$$

$$a = kb + b - 1$$
$$= 2b - 1$$

running time of Euclid's algorithm

Theorem: Suppose that Euclid's algorithm takes n steps for $\gcd(a,b)$ with $a > b$, then $a \geq f_{n+1}$. $\geq 2^{\frac{n+1}{2}-1}$

$= 2^{\frac{n-1}{2}}$.

Proof:

Set $r_{n+1} = a, r_n = b$ then Euclid's algorithm computes

$$a \quad r_{n+1} = q_n r_n + r_{n-1}$$

$$r_n = q_{n-1} r_{n-1} + r_{n-2}$$

$$\vdots$$

$$r_3 = q_2 r_2 + r_1$$

$$r_2 = q_1 r_1 + r_0$$

$f_{n+1} = f_n + f_{n-1}$

each quotient $q_i \geq 1$

$r_1 \geq 1 = f_1$.

$r_0 = 0$

$r_{n+1} \geq r_n + r_{n-1}$

$r_{k+1} \geq r_k + r_{k-1}$

$\geq f_k + f_{k-1} = f_{k+1}$