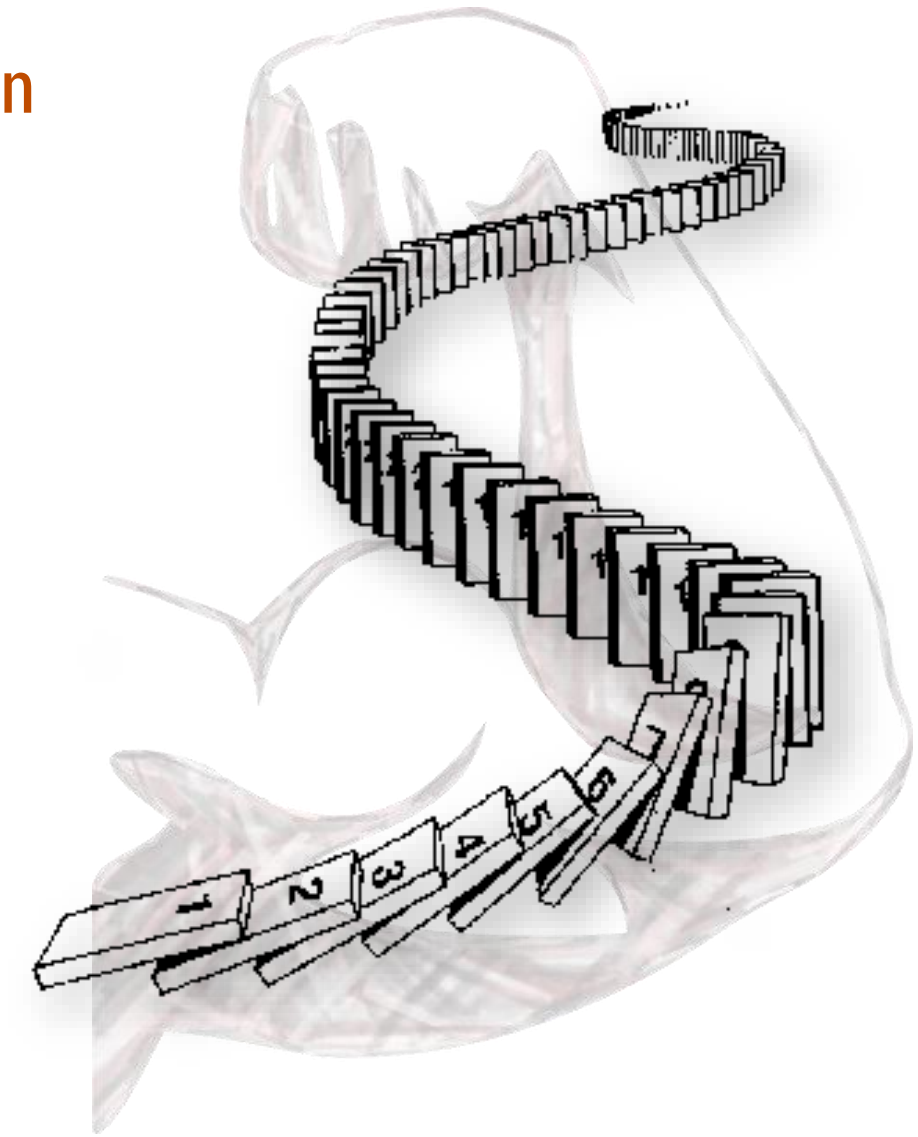
Spring 2015

Lecture 16: **Strong** induction



administrative

Evan covering office hours today [CSE 624, 2:30-3:30pm]

MIDTERM FRIDAY (IN THIS ROOM, USUAL TIME)

- Closed book.
- One page (front and back) of hand-written notes allowed.
- Exam includes induction and strong induction!
- Homework #5 is up now, but due on Friday, May 15th.

Review sessions:

- **James Wednesday @ 6PM** [probably in EEB 105]
- **Additional: …?**

review: **induction** is a rule of inference

Domain: Natural Numbers

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \rightarrow P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

review: using the induction rule in a formal proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\ (P(k)\ \rightarrow\ P(k+1)) \end{array}}{\therefore\ \forall\, n\ P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. P(k) $\rightarrow$ P(k+1) — Direct Proof Rule
7. $\forall$ k (P(k) $\rightarrow$ P(k+1)) — Intro $\forall$ from 2-6
8. $\forall$ n P(n) — Induction Rule 1&7

review: format of an induction proof

$$P(0)$$
$$\forall\, k\ (P(k)\ \rightarrow\ P(k+1))$$

$$\therefore\ \forall\, n\ P(n)$$

1. Prove P(0) — **Base Case**
2. Let k be an arbitrary integer ≥ 0
3. Assume that P(k) is true — **Inductive Hypothesis**
4. ...
5. Prove P(k+1) is true — **Inductive Step**
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

Conclusion

review: inductive proof in five easy steps

Proof:

1. "We will show that P(n) is true for every n ≥ 0 by **induction**."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

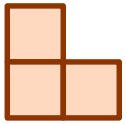
   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:

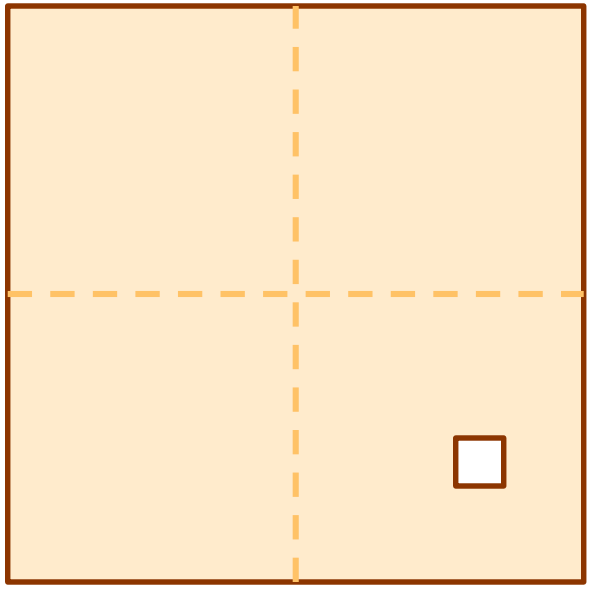
   Use the goal to figure out what you need.

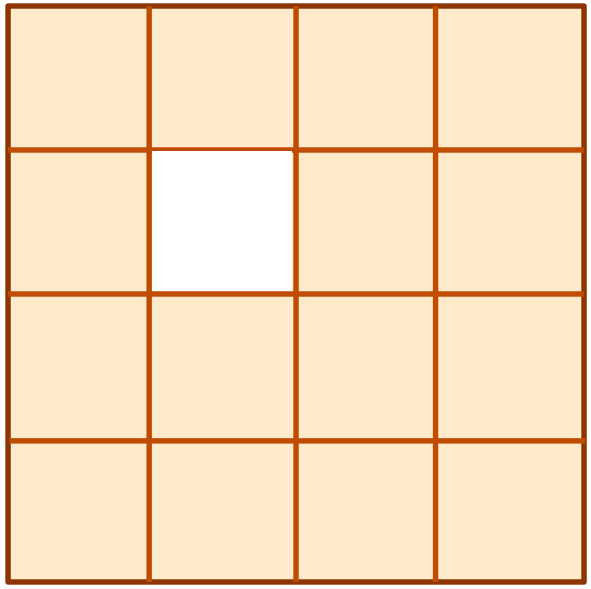
Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !)

5. "Conclusion: Result follows by induction."

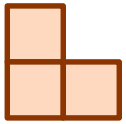
checkerboard tiling

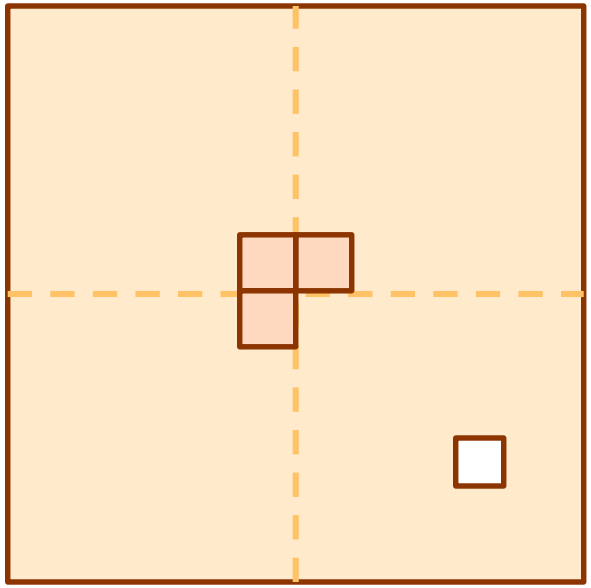
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 

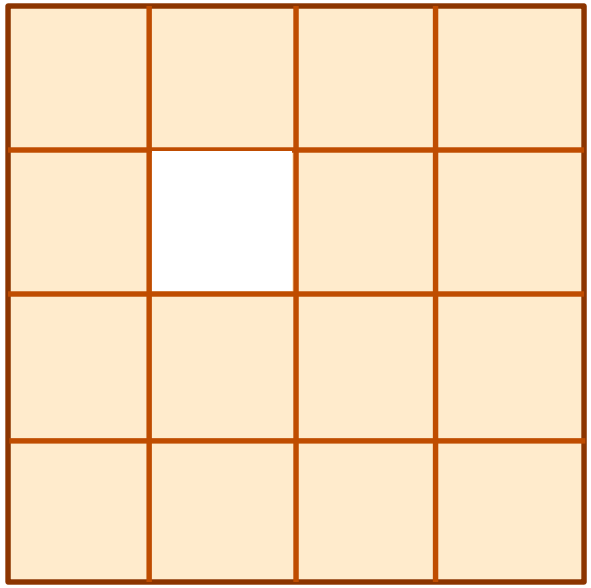




checkerboard tiling

Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 





Place piece in the center and recurse on each quadrant!

checkerboard tiling

Inductive proof:

$P(n) =$ "A $2^n \times 2^n$ checkboard with one piece removed can be tiled by  pieces.

Base case (Zen): $n = 0$. An empty board can be tiled with no pieces $\Rightarrow P(0)$

Alternate base case: $n = 1$. A 2×2 board with one square missing can be tiled with one piece $\Rightarrow P(1)$

Inductive hypothesis: Assume $P(k)$ for some $k \geq 0$.

Inductive step: Consider any $2^{k+1} \times 2^{k+1}$ board with one square missing.

There exists a way to place a piece in the center so that each quadrant is a $2^k \times 2^k$ board with one square missing.
By IH, there is a way to tile each of those four boards.
Thus we can tile the $2^{k+1} \times 2^{k+1}$ board as well.
We conclude that $P(k+1)$ holds.

By induction, $P(n)$ holds for every $n \geq 1$ (or $n \geq 0$ if we started there).

prove $3^n \geq n^2$ for all $n \geq 3$.

Let P(n) be "$3^n \geq n^2$" for all $n \geq 3$.

We go by induction on n.

Base Case:

$3^3 = 27 \geq 9 = 3^2$. So, P(3) is true.

Induction Hypothesis:

Suppose P(k) is true for some arbitrary $k \geq 3$.

Induction Step:

Note that $3^{k+1} = 3(3^k) \geq 3(k^2)$, by the IH.

Furthermore, note that $(k+1)^2 = k^2 + 2k + 1$.

Note that since $k \geq 3$, $k^2 \geq 3k \geq 2k$. And similarly, $k^2 \geq 1$.

So, continuing from above:

$$3^{k+1} = 3(3^k) \geq 3(k^2) = k^2 + k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$$

Since this is exactly P(k+1), we've shown $P(k) \rightarrow P(k+1)$

Thus, P(n) is true for all $n \geq 3$, by induction.

prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Note that $2(n+1)^3 = 2n^3 + 6n^2 + 6n + 2$.

Let P(n) be "$2n^3 + 2n - 5 \geq n^2$" for all $n \geq 2$.

We go by induction on n.

Base Case:

$2*2^3 + 2*2 - 5 = 45 \geq 4 = 2^2$. So, P(0) is true.

Induction Hypothesis:

Suppose P(n) is true for some arbitrary $n \geq 2$.

Induction Step: Then, note that…

$$
\begin{aligned}
(n+1)^2 &\leq n^2 + 2n + 1 \\
&\leq (2n^3 + 2n - 5) + 2n + 1 && \text{(by IH)} \\
&\leq (2n^3 + 4n + 1) - 5 && \text{(Re-arranging)} \\
&\leq (2n^3 + 6n^2 + 6n + 2) - 5 && (4n + 1 \leq 6n + 6n^2 + 2) \\
&\leq 2(n+1)^3 - 5 && \text{(Factoring)} \\
&\leq 2(n+1)^3 + 2n - 5 && (0 \leq 2n)
\end{aligned}
$$

Since this is exactly P(k+1), we've shown P(k) → P(k+1)

Thus, P(n) is true for all $n \geq 3$, by induction.

strong induction

$$P(0)$$

$$\forall k \left(\left(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k) \right) \rightarrow P(k+1) \right)$$

$$\therefore \forall n \, P(n)$$

Follows from ordinary induction applied to
$Q(n) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(n)$

strong induction English proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

every integer at least 2 is the product of primes

We argue by strong induction.

$P(n)$ = "n can be expressed as a product of primes" for $n \geq 2$.

Base Case:

Note that 2 is prime; so, we can express it as "2" which is a product of primes.

Induction Hypothesis:

Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ is true for some $k \geq 2$.

Induction Step:

We go by cases.

Suppose k+1 is prime. Then, "k+1" is a product of primes.

Suppose k+1 is composite. Then, $k+1 = ab$ for some a and b such that $1 < a, b < k+1$.

By our IH, we know $a = p_1 p_2 \cdots p_m$ and $b = q_1 q_2 \cdots q_n$.

So, $k+1 = ab$ = "$p_1 p_2 \cdots p_m q_1 q_2 \cdots q_n$", which is a product of primes.

Thus, our claim is true for $n \geq 2$ by strong induction.

recursive definition of functions

- $F(0) = 0;\ F(n + 1) = F(n) + 1$ **for all** $n \geq 0$
- $G(0) = 1;\ G(n + 1) = 2 \times G(n)$ **for all** $n \geq 0$
- $0! = 1;\ (n + 1)! = (n + 1) \times n!$ **for all** $n \geq 0$
- $H(0) = 1;\ H(n + 1) = 2^{H(n)}$ **for all** $n \geq 0$

Fibonacci numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \quad \textbf{for all } n \geq 2$$

bounding the Fibonacci numbers

Theorem: $f_n < 2^n$ **for all** $n \geq 2$.

$P(n) =$ "$f_n < 2^n$"

Base case: $f_2 = f_1 + f_0 = 1 < 4 = 2^2 \implies P(2)$

(Strong) IH: Assume $P(2), P(3), \ldots, P(n)$

Inductive step: $f_{n+1} = f_n + f_{n-1}$

$$< 2^n + 2^{n-1} \qquad \text{(IH)}$$

$$< 2 \cdot 2^n = 2^{n+1}$$

So by induction, $P(n)$ $\forall n \geq 2$.

$\square$