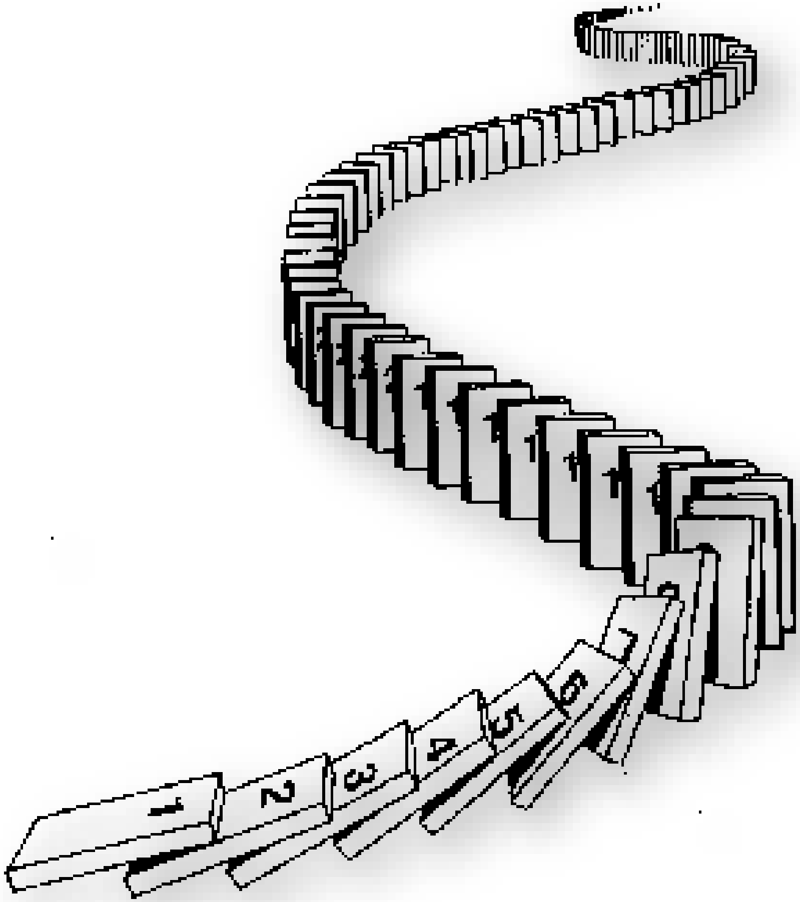
Spring 2015

# Lecture 15: Induction



# prove: for all $n > 0$, $a$ is odd $\rightarrow a^n$ is odd

Let $n > 0$ be arbitrary.

Suppose that $a$ is odd. We know that if $a, b$ are odd, then $ab$ is also odd.

$a$ odd $\rightarrow a \cdot a = a^2$ odd $\rightarrow a \cdot a^2 = a^3$ odd $\rightarrow \ldots$

So: $(\cdots((a \cdot a) \cdot a) \cdot \cdots \cdot a) = a^n$ [$n$ times]

$\forall n \geq 0$ $a^n$ is odd

Those "$\cdots$"s are a problem! We're trying to say "we can use the same argument over and over…"

We'll come back to this.

# mathematical induction

## Method for proving statements about all integers ≥ 0

- A new logical inference rule!
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- Particularly useful for reasoning about programs!

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

Assume $f(x-1)=x-1$

$\Downarrow$

$f(x)=x$

```
public int f(int x) {
    if (x == 0) { return 0; }
        else { return f(x–1)+1; }}
```

$f(0)=0$

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

$\Longrightarrow \forall x \; f(x)=x$

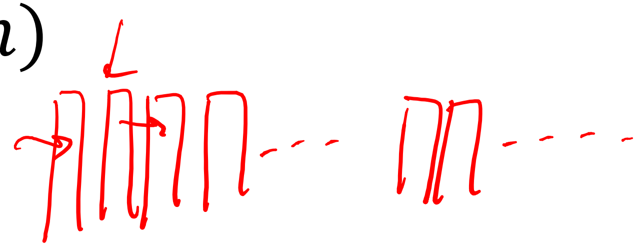
# **induction** is a rule of inference

Domain: Natural Numbers

$$\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \rightarrow P(k+1)) \\ \hline \therefore \forall\, n\, P(n) \end{array}$$

# using the induction rule in a formal proof

$$\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \;\rightarrow\; P(k+1)) \\ \hline \therefore\ \forall\, n\, P(n) \end{array}$$



1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

$\forall n \geq s \; P(n)$

# format of an induction proof

$Q(k) = P(k+s)$

$$P(0)$$
$$\forall\, k\, (P(k) \;\rightarrow\; P(k+1))$$
$$\overline{\therefore \forall\, n\, P(n)}$$

$P(s)$

$\forall k\, (P(k) \rightarrow P(k+1))$

$\therefore \forall n \geq s \; P(n)$

1. Prove P(0)

**Base Case**

2. Let k be an arbitrary integer ≥ 0
3. Assume that P(k) is true

**Inductive Hypothesis**

4. ...
5. Prove P(k+1) is true

**Inductive Step**

| | |
|---|---|
| 6. $P(k) \rightarrow P(k+1)$ | Direct Proof Rule |
| 7. $\forall$ k (P(k) $\rightarrow$ P(k+1)) | Intro $\forall$ from 2-6 |
| 8. $\forall$ n P(n) | Induction Rule 1&7 |

**Conclusion**

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

$$1 + 2 + \cdots + 2^n = 2^{n+1} - 1$$

Can we describe the pattern?

Yes.

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$$

# proving $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$

$P(n) =$ "$1 + 2 + \cdots + 2^n = 2^{n+1} - 1$"

Base case: $P(0)$ "$1 = 2^{0+1} - 1$"

$1 = 1$

$P(0)$ holds.

IH: Assume $P(k)$ holds for some (arbitrary) $k \geq 0$

Ind step: By $P(k)$, $1 + 2 + \cdots + 2^k = 2^{k+1} - 1$

$\Rightarrow 1 + 2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

$\boxed{\text{Induction}}$ $= 2 \cdot 2^{k+1} - 1$

$= 2^{k+2} - 1$

$\Rightarrow P(k+1)$

$P(k) \to P(k+1)$ ... $\forall k\ P(k) \to P(k+1)$ $\forall n \geq 0\ P(n)$

# inductive proof in five easy steps

**Proof:**

1. "We will show that P(n) is true for every n ≥ 0 by **induction**."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !)
5. "Conclusion: Result follows by induction."

# proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

1. Let P(n) be "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". We will show P(n) is true for all natural numbers by induction.
2. Base Case (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$
3. Induction Hypothesis: Suppose that P(k) is true for some arbitrary $k \geq 0$.
4. Induction Step:

   Goal: Show P(k+1), i.e. show $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

   $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ by IH

   Adding $2^{k+1}$ to both sides, we get:

   $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly P(k+1).
5. Thus P(k) is true for all $k \in \mathbb{N}$, by induction.

# another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- $\dots$

# prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

$P(n) =$ " $3 \mid 2^{2n} - 1$ "

$\boxed{\text{Goal: } \forall_{n \geq 0}\, P(n)}$

Base case: $2^{2 \cdot 0} - 1 = 0 = 0 \cdot 3$ $\implies P(0)$

$2^{2(k+1)} - 1 = 3h$

Ind. Hypoth: Assume $P(k)$ holds for some $k \geq 0$

$2^{2k} - 1 = \ldots$

$3 \mid 2^{2k} - 1 \quad \left( \overset{?}{\iff} 3 \mid 2^{2(k+1)} - 1 \right)$

$\implies 2^{2k} - 1 = 3j$ for some $j$

$\implies 4(2^{2k} - 1) = 12j$

$\implies 2^{2(k+1)} - 4 = 12j$

$\implies 2^{2(k+1)} - 1 = 12j + 3$

$= 3(4j+1) \implies P(k+1)$

$4(2^{2k} - 1)$

$= 2^{2k+2} - 4$

$= 2^{2(k+1)} - 1 - 3$

$2^{2(k+1)} - 1 = 3 + 4(2^{2k} - 1)$

## For all $n \geq 1$: $1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$P(n) =$ " $1 + \cdots + n = \frac{n(n+1)}{2}$ "

We will prove $\forall n \geq 1$ $P(n)$ by induction.

Base case: $P(1)$ holds b/c

$$1 = \frac{1 \cdot (1+1)}{2}.$$

Ind Hypoth: Assume $1 + \cdots + k = \frac{k(k+1)}{2}$

Ind step: From (IH), $1 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$

$$= (k+1)\left(\frac{k}{2} + 1\right)$$

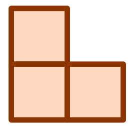
$$= (k+1)\left(\frac{k+2}{2}\right)$$

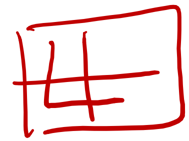
$$= \frac{(k+1)(k+2)}{2}$$

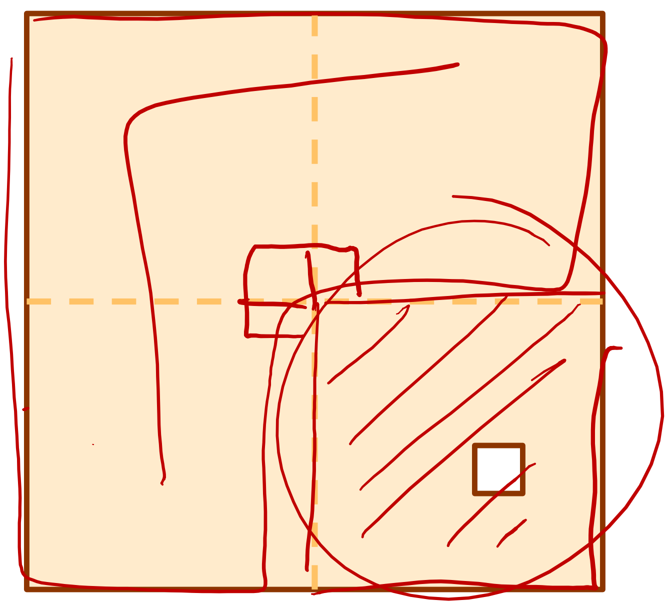
$\Rightarrow P(k+1)$

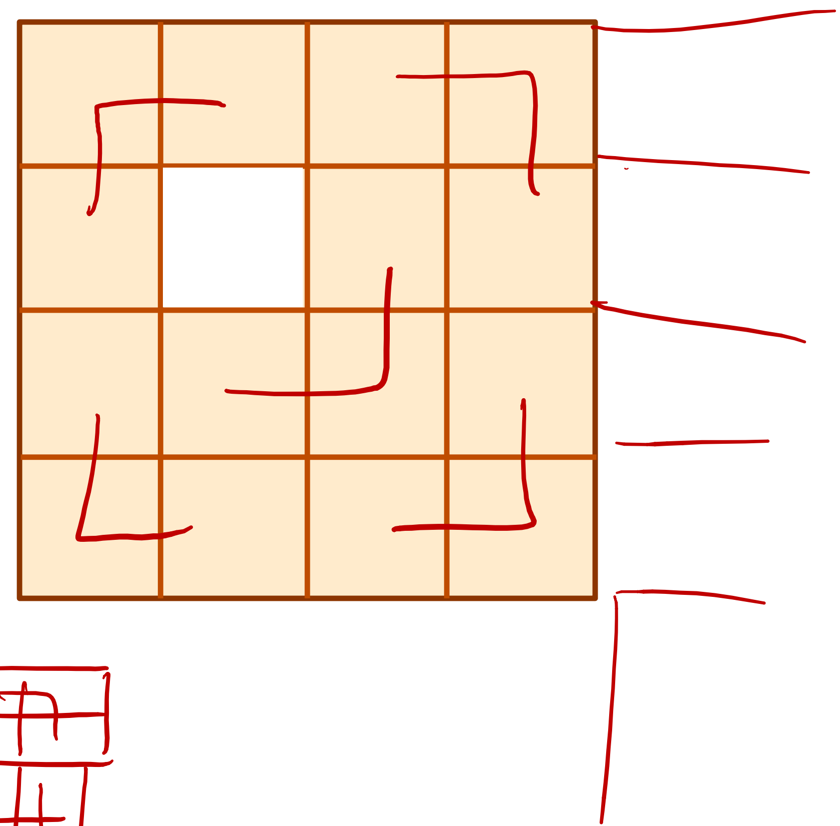
By induction, $\forall n \geq 1$ $P(n)$ holds.

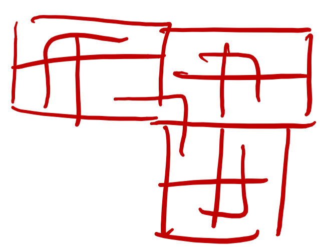
# checkerboard tiling

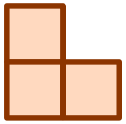
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 









Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 

## prove: $n^n \geq n!$ for all $n \geq 1$