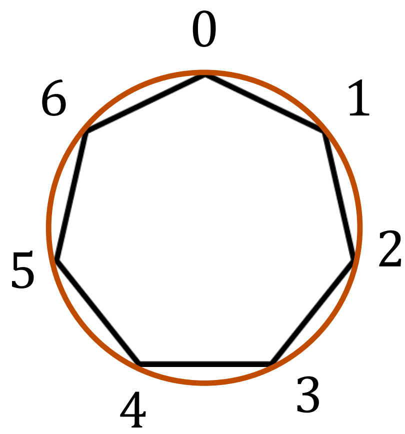
# Spring 2015

## Lecture 11: Modular arithmetic and applications

# arithmetic mod 7

$a +_7 b = (a + b) \bmod 7$

$a \times_7 b = (a \times b) \bmod 7$




| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

Integers a, b, with $a \neq 0$. We say that a **divides** b iff there is an integer k such that $b = k\,a$. The notation $a \mid b$ denotes "a divides b."

# review: division theorem

Let $a$ be an integer and $d$ a positive integer. Then there are *unique* integers $q$ and $r$, with $0 \le r < d$, such that $a = d\, q + r$.

$q = a \textbf{ div } d$ $\qquad\qquad$ $r = a \textbf{ mod } d$

Note: $r \ge 0$ even if $a < 0$.
Not quite the same as **`a % d`**.

# modular congruence

Let a and b be integers, and m be a positive integer.
We say *a* is **congruent** to *b* **modulo** *m* if *m* divides *a* – *b*.
We use the notation a ≡ b (mod m) to indicate that a is congruent to b modulo m.

**Theorem:** Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

**Proof:** $\Rightarrow$

Suppose that $a \equiv b \pmod{m}$.

By definition: $a \equiv b \pmod{m}$ implies $m \mid (a - b)$

which by definition implies that $a - b = km$ for some integer k.

Therefore $a = b + km$.

Taking both sides modulo m we get

$a \bmod m = (b+km) \bmod m = b \bmod m$

**Theorem:** Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if a mod m = b mod m.

**Proof:** $\Leftarrow$

Suppose that a mod m = b mod m.

By the division theorem, a = mq + (a mod m) and
b = ms + (b mod m) for some integers q,s.

$$
\begin{aligned}
a - b &= (mq + (a \bmod m)) - (mr + (b \bmod m)) \\
&= m(q - r) + (a \bmod m - b \bmod m) \\
&= m(q - r) \quad \text{since } a \bmod m = b \bmod m
\end{aligned}
$$

Therefore m | (a-b) and so $a \equiv b \pmod{m}$

# consistency of addition

Let $m$ be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{a + c \equiv b + d \pmod{m}}$

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

Unrolling definitions gives us some $k$ such that $a - b = km$, and some $j$ such that $c - d = jm$.

Adding the equations together gives us $(a + c) - (b + d) = m(k + j)$. Now, re-applying the definition of mod gives us $a + c \equiv b + d \pmod{m}$.

# consistency of multiplication

Let $m$ be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{ac \equiv bd \pmod{m}}$

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

Unrolling definitions gives us some $k$ such that $a - b = km$, and some $j$ such that $c - d = jm$.

Then, $a = km + b$ and $c = jm + d$.
Multiplying both together gives us

$$ac = (km + b)(jm + d) = kjm^2 + kmd + jmb + bd$$

Rearranging gives us $ac - bd = m(kjm + kd + jb)$.
Using the definition of mod gives us $ac \equiv bd \pmod{m}$.

Let $n$ be an integer.
Prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

Let $n$ be an integer.
Prove that $n^2 \equiv 0 \pmod{4}$ or $n^2 \equiv 1 \pmod{4}$

**Case 1 (n is even):**

Suppose $n \equiv 0 \pmod{2}$.
Then, $n = 2k$ for some integer k.
So, $n^2 = (2k)^2 = 4k^2$.
So, by definition of congruence, $n^2 \equiv 0 \pmod{4}$.

**Case 2 (n is odd):**

Suppose $n \equiv 1 \pmod{2}$.
Then, $n = 2k + 1$ for some integer k.
So, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$.
So, by definition of congruence, $n^2 \equiv 1 \pmod{4}$.

# n-bit unsigned integer representation

- Represent integer x as sum of powers of 2:
  If $x = \sum_{i=0}^{n-1} b_i 2^i$ where each $b_i \in \{0,1\}$
  then representation is $b_{n-1} \cdots b_2 \, b_1 \, b_0$

  99 = 64 + 32 + 2 + 1
  18 = 16 + 2

- For n = 8:

  99: 0110 0011
  18: 0001 0010

# sign-magnitude integer representation

n-bit signed integers

Suppose $-2^{n-1} < x < 2^{n-1}$

First bit as the sign, n-1 bits for the value

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

99: 0110 0011

-18: 1001 0010

Any problems with this representation?

# two's complement representation

n-bit signed integers, first bit will still be the sign bit

Suppose $0 \leq x < 2^{n-1}$,
 $x$ is represented by the binary representation of $x$
Suppose $0 \leq x \leq 2^{n-1}$,
 $-x$ is represented by the binary representation of $2^n - x$

**Key property:** Two's complement representation of any number y is equivalent to y mod $2^n$ so arithmetic works mod $2^n$

99 = 64 + 32 + 2 + 1
18 = 16 + 2

For n = 8:
 99: 0110 0011
 -18: 1110 1110

# sign-magnitude vs. two's complement

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1111 | 1110 | 1101 | 1100 | 1011 | 1010 | 1001 | 0000 | 0001 | 0010 | 0011 | 0100 | 0101 | 0110 | 0111 |

Sign-Magnitude

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 | 0000 | 0001 | 0010 | 0011 | 0100 | 0101 | 0110 | 0111 |

Two's complement

# two's complement representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $2^n - x$

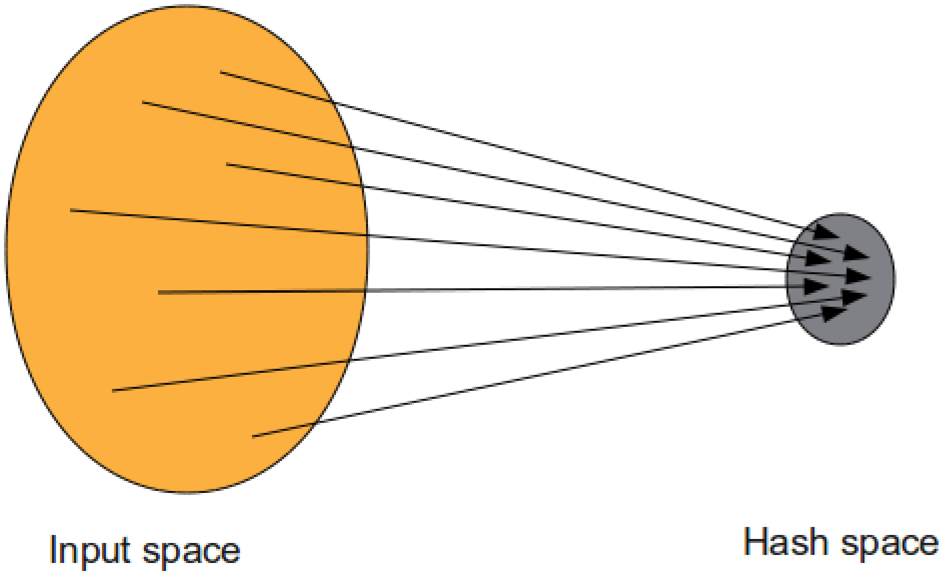
- To compute this: Flip the bits of $x$ then add 1:
  – All 1's string is $2^n - 1$, so
    Flip the bits of $x$ $\equiv$ replace $x$ by $2^n - 1 - x$

# basic applications of mod

- Hashing
- Pseudo random number generation
- Simple cipher

## Scenario:

Map a small number of data values from a large domain $\{0, 1, \ldots, M-1\}$ into a small set of locations $\{0,1, \ldots, n-1\}$ so one can quickly check if some value is present.



## Scenario:

Map a small number of data values from a large domain $\{0, 1, \ldots, M-1\}$ into a small set of locations $\{0,1, \ldots, n-1\}$ so one can quickly check if some value is present

- $\mathrm{hash}(x) = x \bmod p$ for $p$ a prime close to $n$
  - or $\mathrm{hash}(x) = (ax + b) \bmod p$
- Depends on all of the bits of the data
  - helps avoid collisions due to similar values
  - need to manage them if they occur

# pseudo-random number generation

Linear Congruential method:

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random $x_0, a, c, m$ and produce a long sequence of $x_n$'s

[good for some applications, really bad for many others]

# simple ciphers

- Caesar cipher, A = 1, B = 2, . . .
  - HELLO WORLD
- Shift cipher
  - $f$(p) = (p + k) mod 26
  - $f^{-1}$(p) = (p – k) mod 26
- More general
  - $f^{-1}$(p) = (ap + b) mod 26

# modular exponentiation mod 7

| x | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | |

| a | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | |

# modular exponentiation mod 7

| x | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 |

| a | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | |

# modular exponentiation mod 7

| x | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 |

| a | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 4 | 1 | 2 | 4 | 1 |
| 3 | 3 | 2 | 6 | 4 | 5 | 1 |
| 4 | 4 | 2 | 1 | 4 | 2 | 1 |
| 5 | 5 | 4 | 6 | 2 | 3 | 1 |
| 6 | 6 | 1 | 6 | 1 | 6 | 1 |