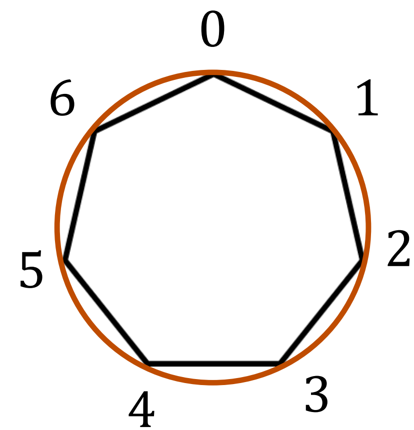
Spring 2015

Lecture 11: Modular arithmetic and applications

arithmetic mod 7

$$a +_7 b = (a + b) \bmod 7$$

$$a \times_7 b = (a \times b) \bmod 7$$



+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Integers a, b, with $a \neq 0$. We say that a **divides** b iff there is an integer k such that $b = k\,a$. The notation $a \mid b$ denotes "a divides b."

$a \mid b \iff$

b is an integer mult of a.

$$\boxed{a \mid b \iff \frac{b}{a}}$$

Let *a* be an integer and *d* a positive integer. Then there are *unique* integers *q* and *r*, with $0 \le r < d$, such that $a = d\,q + r$.

$q = a$ **div** d $r = a$ **mod** d

q r

$d \,|\, a$

$0 \le r < d$

$d = 5$ $q = 4$

$a = 22$ $r = 2$

$\cancel{q = -3}$

$d = 5$ $q = -4$

$a = -17$ $r = 3$

Note: r ≥ 0 even if a < 0.
Not quite the same as **`a % d`**.

$\cancel{r = -2}$

modular congruence

Let a and b be integers, and m be a positive integer.
We say *a* is **congruent** to *b* **modulo** *m* if *m* divides *a* – *b*.
We use the notation a ≡ b (mod m) to indicate that a is congruent to b modulo m.

$7 \equiv 2 \pmod{5}$

$a \equiv b \pmod{m}$

$\Longleftrightarrow$

$m \mid a-b$

$7 \equiv -3$

$\equiv -8 \pmod{5}$

$\equiv -13$

$\equiv 2 \pmod{5}$

0, 1, 2, 3, 4

Theorem: Let a and b be integers, and let m be a positive integer. Then a ≡ b (mod m) if and only if a mod m = b mod m.

$a \bmod m$ = canonical representative

$$\boxed{a \equiv b \pmod{m} \iff a \bmod m = b \bmod m}$$

$a \equiv b \pmod{m} \implies m \mid a-b$

$\implies \exists k$ s.t.

$a - b = km$

$\implies a = b + km$

$\implies a \bmod m = b \bmod m$ ✓

$m = 5$

$$\overset{3}{(a} + \overset{3}{b)} \bmod m \neq a \bmod m + b \bmod m$$

Theorem: Let a and b be integers, and let m be a positive integer. Then a ≡ b (mod m) if and only if a mod m = b mod m.

$$a \bmod m = b \bmod m \overset{?}{\Longrightarrow} a \equiv b \pmod m$$

$$\Longleftrightarrow$$

$$m \mid a-b$$

$$\Longleftrightarrow$$

$$a-b = km$$

for some k.

$$a = m(a \operatorname{div} m) + (a \bmod m)$$

$$b = m(b \operatorname{div} m) + (b \bmod m)$$

$$a-b = m((a \operatorname{div} m) - (b \operatorname{div} m)) + (a \bmod m - b \bmod m)$$

$$= m\underbrace{(a \operatorname{div} m - b \operatorname{div} m)}_{\text{integer}}$$

✓

$(a+b) \bmod m$

$A \equiv_m B$

consistency of addition

Let m be a positive integer. If a ≡ b (mod m) and c ≡ d (mod m), then **a + c ≡ b + d (mod m)**

$a \equiv b \pmod{m}$ $\Longleftrightarrow$ $m \mid a-b$

$c \equiv d \pmod{m}$ $\quad$ $m \mid c-d$

$\Longleftrightarrow$ $a-b = km$

$c-d = jm$

for some k, j integers

$\Longleftarrow$

$a+c-b-d$

$= (k+j)m$

$\Downarrow$

$m \mid a+c-b-d$

$\Longleftrightarrow$ $a+c \equiv b+d \pmod{m}$

$\Longleftarrow$

consistency of multiplication

> Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{ac \equiv bd \pmod{m}}$

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

Unrolling definitions gives us some k such that $a - b = km$, and some j such that $c - d = jm$.

$ac \equiv \cancel{m(\ldots)} + K \pmod{m}$

Then, $a = km + b$ and $c = jm + d$.

Multiplying both together gives us

$$ac = (km + b)(jm + d) = kjm^2 + kmd + jmb + bd$$

Rearranging gives us $ac - bd = m(kjm + kd + jb)$.

Using the definition of mod gives us $ac \equiv bd \pmod{m}$.

$\left[n^2 \equiv ? \pmod 8 \right]$ 0, 1, 4, 5

example

Let n be an integer.
Prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

Case analysis:

$0^2 \equiv 0 \pmod 4 \leftarrow n \equiv 0 \pmod 4$

$1^2 \equiv 1 \pmod 4 \leftarrow n \equiv 1 \pmod 4$

$2^2 \equiv 0 \pmod 4 \leftarrow n \equiv 2 \pmod 4$

$3^2 \equiv 1 \pmod 4 \leftarrow n \equiv 3 \pmod 4$

Pf #2: n even $\Rightarrow n = 2k$ for some int k

$\Rightarrow n^2 = 4k^2$

$\Rightarrow n^2 \equiv 0 \pmod 4$

n odd $\Rightarrow n = 2k+1$ for some int $k \Rightarrow 4k^2 + 4k + 1 = n^2$

$\Rightarrow n^2 \equiv 1 \pmod 4$.

example

Let n be an integer.
Prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

Case 1 (n is even):

Suppose $n \equiv 0 \pmod 2$.
Then, $n = 2k$ for some integer k.
So, $n^2 = (2k)^2 = 4k^2$.
So, by definition of congruence, $n^2 \equiv 0 \pmod 4$.

Case 2 (n is odd):

Suppose $n \equiv 1 \pmod 2$.
Then, $n = 2k + 1$ for some integer k.
So, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$.
So, by definition of congruence, $n^2 \equiv 1 \pmod 4$.

n-bit unsigned integer representation

- Represent integer x as sum of powers of 2:
  If $x = \sum_{i=0}^{n-1} b_i 2^i$ where each $b_i \in \{0,1\}$
  then representation is $b_{n-1} \cdots b_2\, b_1\, b_0$

  99 = 64 + 32 + 2 + 1
  18 = 16 + 2

- For n = 8:

  99: 0110 0011
  18: 0001 0010

sign-magnitude integer representation

0000 0000
1000 0000

n-bit signed integers
Suppose $-2^{n-1} < x < 2^{n-1}$
First bit as the sign, n-1 bits for the value

99 = 64 + 32 + 2 + 1
18 = 16 + 2

$\begin{array}{r} 99 \\ -18 \\ \hline \end{array}$

$\begin{array}{r} 0110\ 0011 \\ 1001\ 0010 \\ \hline 1111\ 0101 \end{array}$

For n = 8:

99: 0110 0011
-18: 1001 0010

18: 001 0010

Any problems with this representation? yes.
(surely)

two's complement representation

n-bit signed integers, first bit will still be the sign bit

Suppose $0 \leq x < 2^{n-1}$,

x is represented by the binary representation of x

Suppose $0 \leq x \leq 2^{n-1}$,

$-x$ is represented by the binary representation of $2^n - x$

99 :

Key property: Two's complement representation of any number y is equivalent to y mod 2^n so arithmetic works mod 2^n

99 = 64 + 32 + 2 + 1
18 = 16 + 2

For n = 8:

99: 0110 0011
-18: 1110 1110

$256 - 18 = 238$

$= 2 + 4 + 8 + 32 + 64 + 128$

sign-magnitude vs. two's complement

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1111	1110	1101	1100	1011	1010	1001	0000	0001	0010	0011	0100	0101	0110	0111

Sign-Magnitude

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111

Two's complement

two's complement representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $2^n - x$

1111 1111 = −1

1111 1111

$-\underline{0 x_7 x_6 \cdots x_2 x_1}$

$\cdots \bar{x}_1$

- To compute this: Flip the bits of x then add 1:
  - All 1's string is $2^n - 1$, so

    Flip the bits of x ≡ replace x by $2^n - 1 - x$

$0 < x \leq 2^{n-1}$

$-x = 1\,(2^n - x)_2$

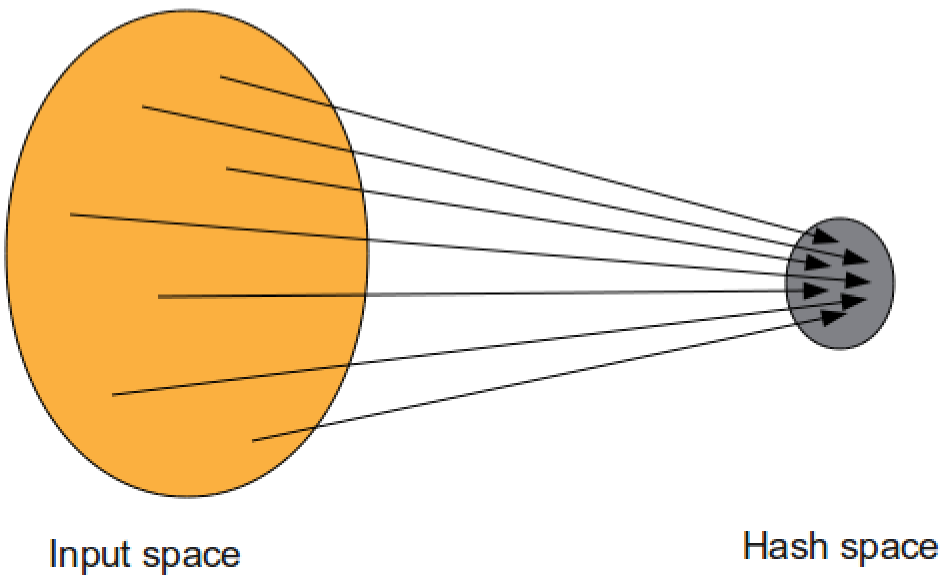
$$\boxed{\text{Flip all bits of } x \to 2^n - 1 - x}$$

basic applications of mod

- Hashing
- Pseudo random number generation
- Simple cipher

Scenario:

Map a small number of data values from a large domain $\{0, 1, \ldots, M-1\}$ into a small set of locations $\{0,1, \ldots, n-1\}$ so one can quickly check if some value is present.



Scenario:

Map a small number of data values from a large domain $\{0, 1, \dots, M-1\}$ into a small set of locations $\{0,1, \dots, n-1\}$ so one can quickly check if some value is present

- $\mathrm{hash}(x) = x \bmod p$ for p a prime close to n
  - or $\mathrm{hash}(x) = (ax + b) \bmod p$

- Depends on all of the bits of the data
  - helps avoid collisions due to similar values
  - need to manage them if they occur

Linear Congruential method:

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random x_0, a, c, m and produce a long sequence of x_n's

[good for some applications, really bad for many others]

- Caesar cipher, A = 1, B = 2, . . .
  - HELLO WORLD
- Shift cipher
  - f(p) = (p + k) mod 26
  - f^{-1}(p) = (p – k) mod 26
- More general
  - f^{-1}(p) = (ap + b) mod 26

modular exponentiation mod 7

x	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

modular exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

modular exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1	1	1	1	1	1	1
2	2	4	1	2	4	1
3	3	2	6	4	5	1
4	4	2	1	4	2	1
5	5	4	6	2	3	1
6	6	1	6	1	6	1