$\forall x\ P(x)$

P(x) is true for every x in the domain

read as "for all x, P of x"

$\exists x\ P(x)$

There is an x in the domain for which P(x) is true

read as "there exists x, P of x"

negations of quantifiers

- ~~no~~t every positive integer is prime

negation

- some positive integer is not prime

Every p.i. is prime.

- prime numbers do not exist

There is a prime #.

- every positive integer is not prime

" "

negations of quantifiers

- $\forall x$ PurpleFruit(x)
  - "All fruits are purple"
  - What is $\neg\forall x$ PurpleFruit(x)
  - "Not all fruits are purple"

Domain:
Fruit

PurpleFruit(x)

- How about $\exists x$ PurpleFruit(x)?
  - "There is a purple fruit"
  - If it's the negation, all situations should be covered by a statement and its negation.
  - Consider the domain {Orange}: Neither statement is true!
  - No.

- How about $\exists x$ $\neg$PurpleFruit(x)?
  - "There is a fruit that isn't purple"
  - Yes.

de Morgan's laws for quantifiers

$$\boxed{\begin{aligned} \neg\forall x\ P(x) &\equiv \exists x\, \neg P(x) \\ \neg\exists x\ P(x) &\equiv \forall x\, \neg P(x) \end{aligned}}$$

$D = \{x_1, x_2, \ldots\}$

$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots$

$\neg\forall x\, P(x) \equiv \neg(P(x_1) \wedge P(x_2) \wedge \cdots)$

$\equiv \neg P(x_1) \vee \neg P(x_2) \vee \cdots$

$\equiv \exists x\, \neg P(x)$

$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots$

de Morgan's laws for quantifiers

$$\boxed{\begin{array}{l} \neg \forall x \; P(x) \equiv \exists x \neg P(x) \\ \neg \exists x \; P(x) \equiv \forall x \neg P(x) \end{array}}$$

Domain = integers

"There is no largest integer."

$$\begin{aligned} & \neg \exists x \; \forall y \; (x \geq y) \\ \equiv \; & \forall x \neg \forall y \; (x \geq y) \\ \equiv \; & \forall x \; \exists y \neg (x \geq y) \\ \equiv \; & \forall x \; \exists y \; (y > x) \end{aligned}$$

"For every integer there is a larger integer."

scope of quantifiers

example: Notlargest(x) $\equiv \exists y \text{ Greater}(y, x)$

$\equiv \exists z \text{ Greater}(z, x)$

$\boxed{f(7)}$

truth value:

doesn't depend on y or z "bound variables"

does depend on x "free variable"

quantifiers only act on free variables of the formula they quantify

$$\underset{D}{\forall} x\ (\underset{D}{\exists} y\ (P(x, y) \rightarrow \underset{D'}{\forall} x\ Q(y, x)))$$

function f(A)) return g(2A).

let z = 7

function g(A))

let B = A + 3 + z;

return B

scope of quantifiers

$\exists x\ (P(x) \land Q(x))$ **vs.** $(\exists x\ P(x)) \land (\exists x\ Q(x))$

D = pos. integers

$P(x)$ = "$x < 3$"

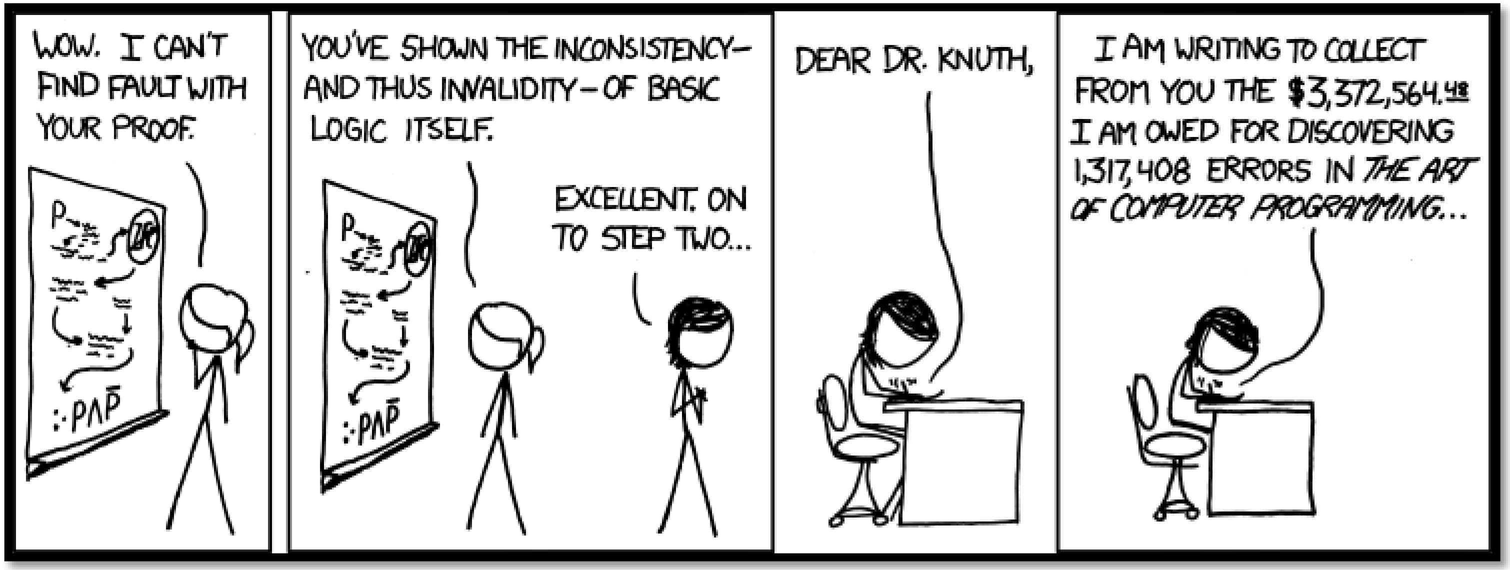
$Q(x)$ = "$x \geq 5$"

$\exists x\ P(x) \land \exists y\ Q(y)$

$P(x)$ = "$x \geq 3$"

Spring 2015

Lecture 6: Predicate Logic, Logical Inference



- Bound variable names don't matter

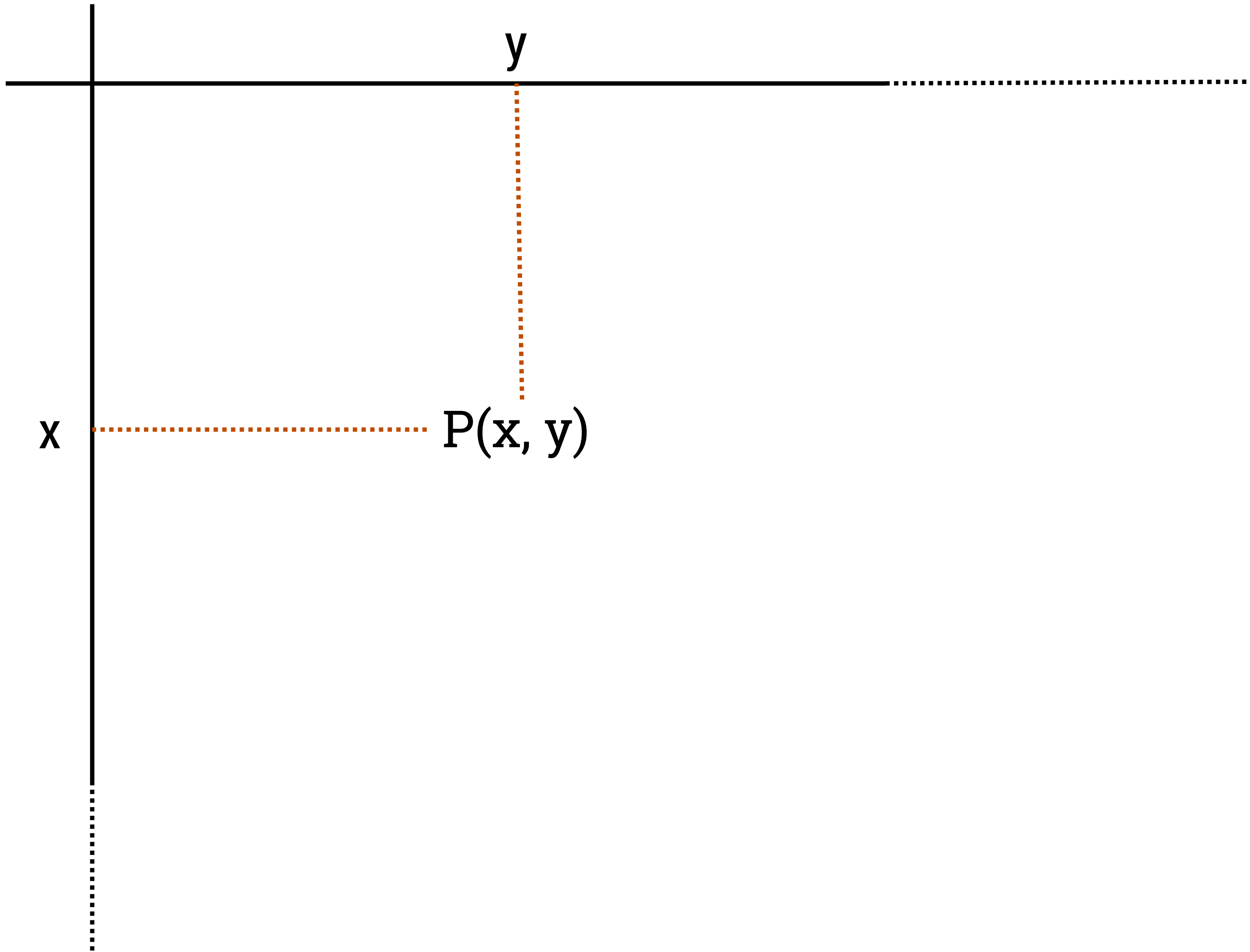
  $$\forall x \exists y\, P(x, y) \equiv \forall a \exists b\, P(a, b)$$

- Positions of quantifiers can sometimes change

  $$\forall x\, (Q(x) \land \exists y\, P(x, y)) \equiv \forall x \exists y\, (Q(x) \land P(x, y))$$

- But: order is important...

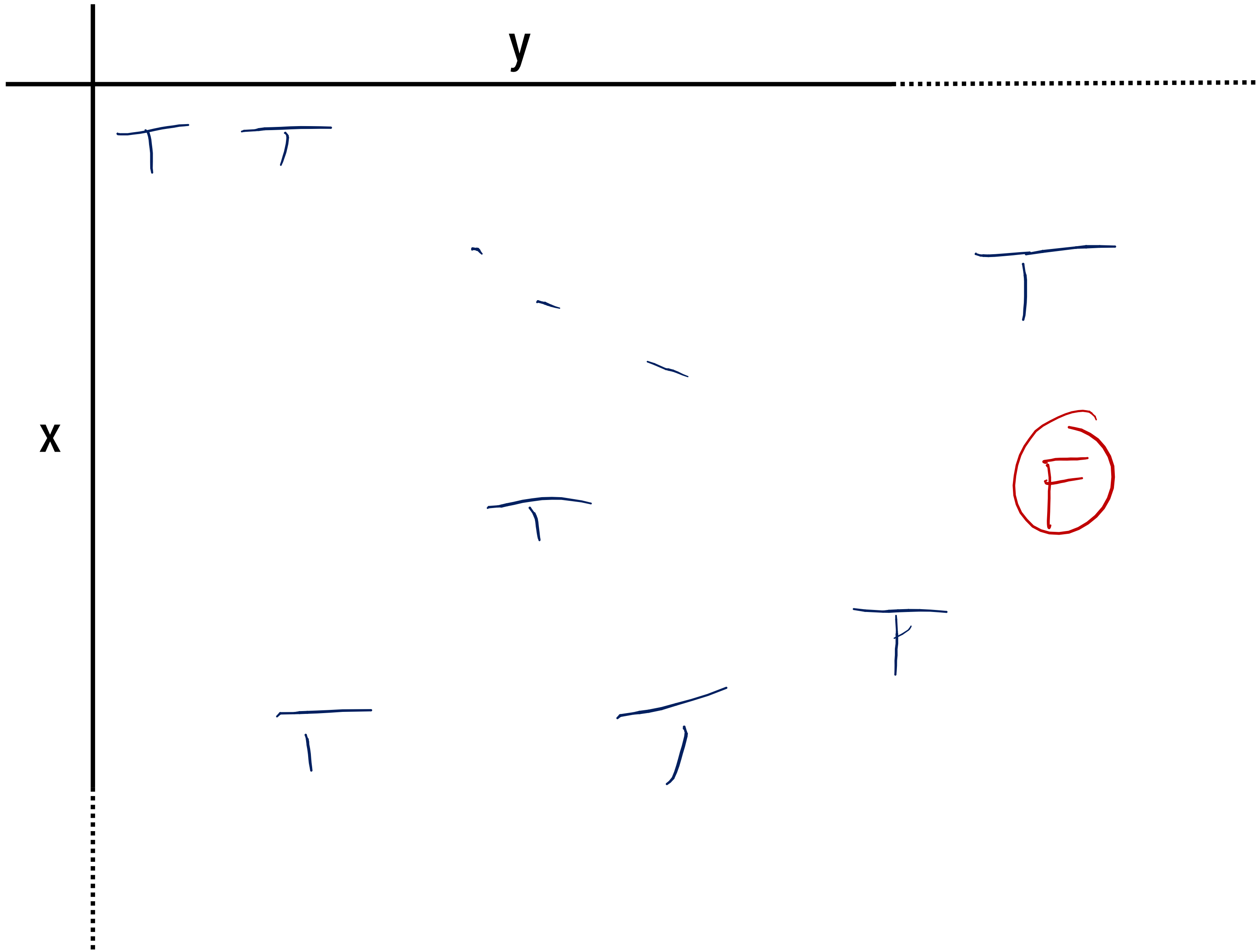
predicate with two variables



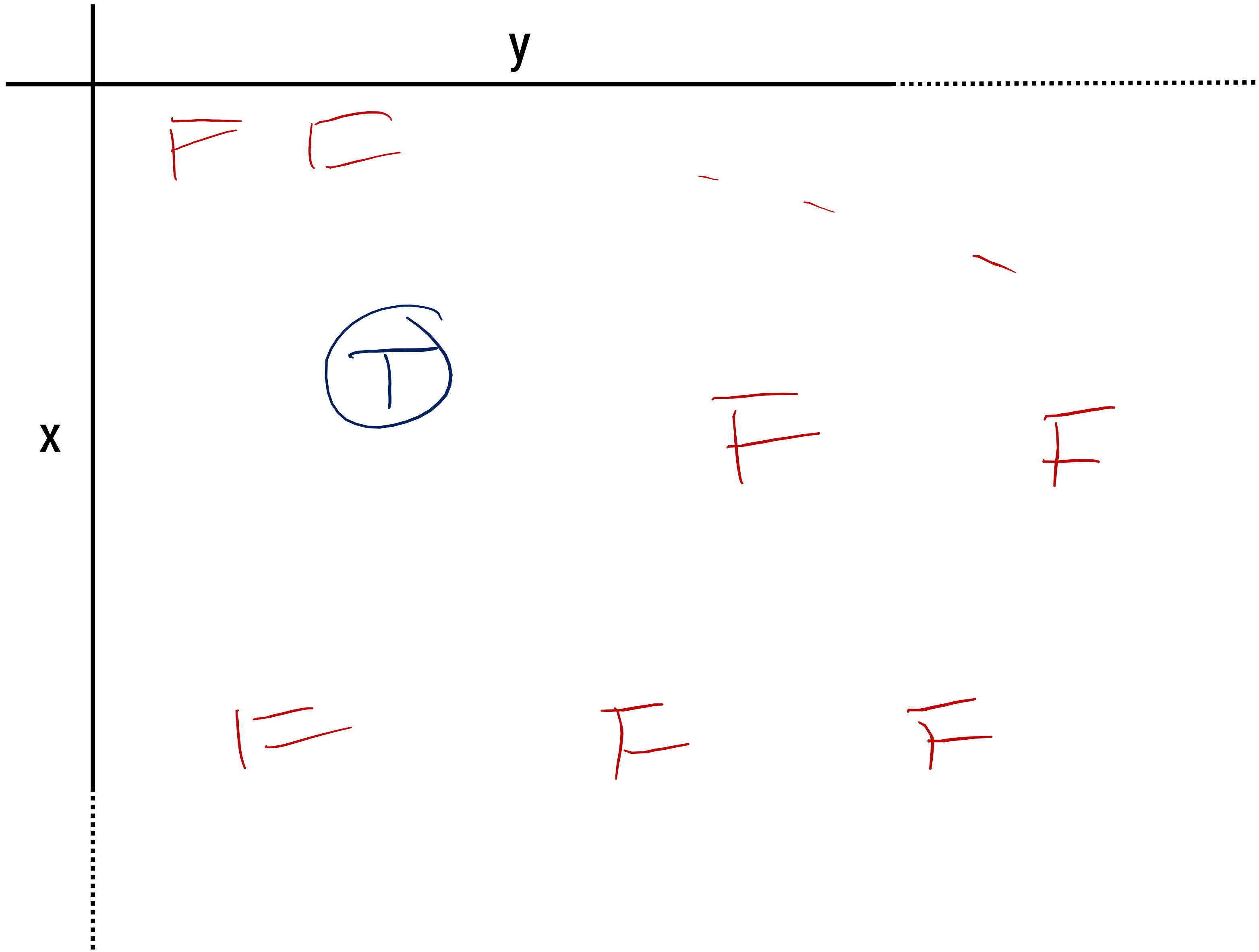
quantification with two variables

expression	**when true**	**when false**
$\forall x \forall y P(x, y)$		
$\exists x \exists y P(x, y)$		
$\forall x \exists y P(x, y)$		
$\exists x \forall y P(x, y)$		

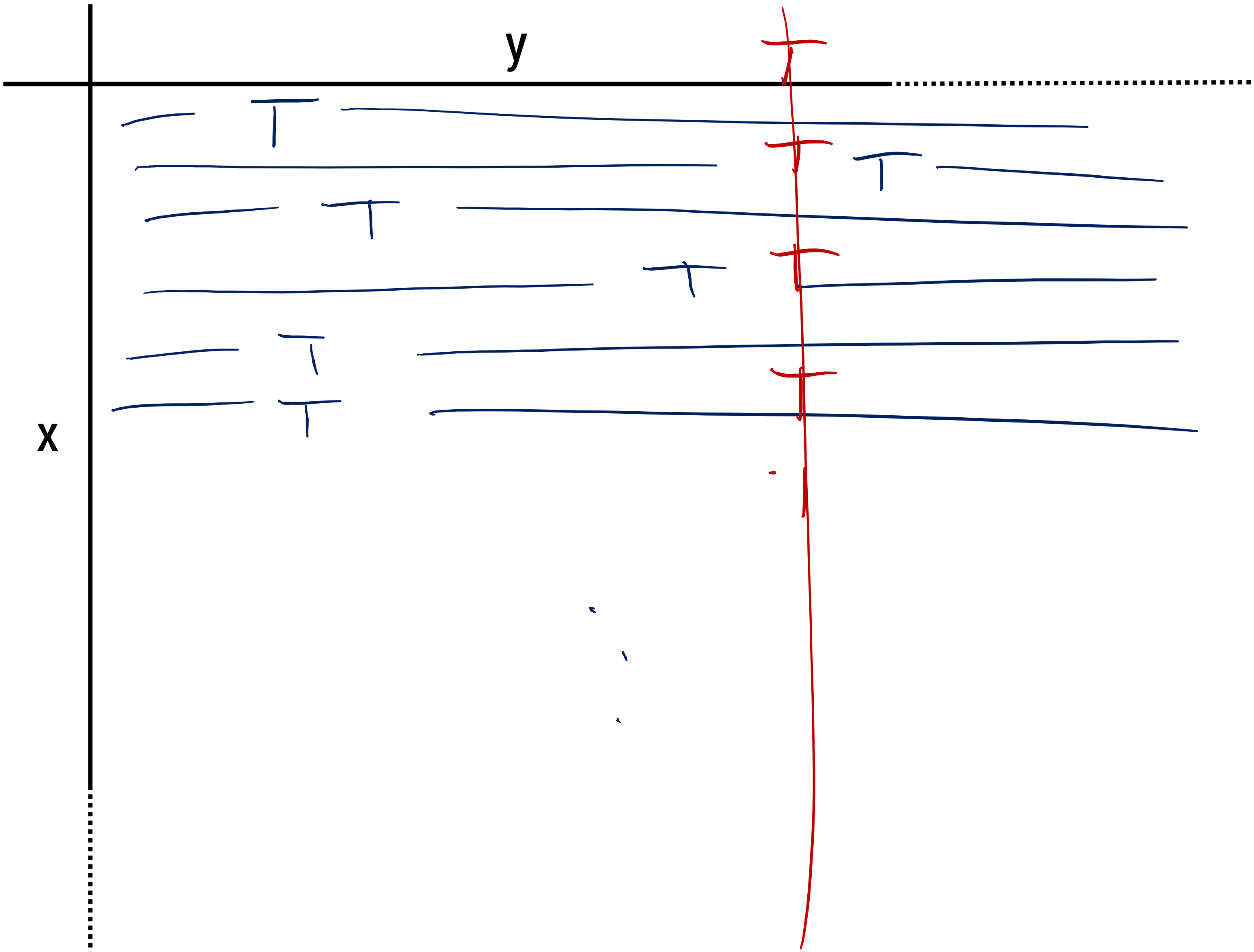
$\forall x \, \forall y \, P(x, y)$



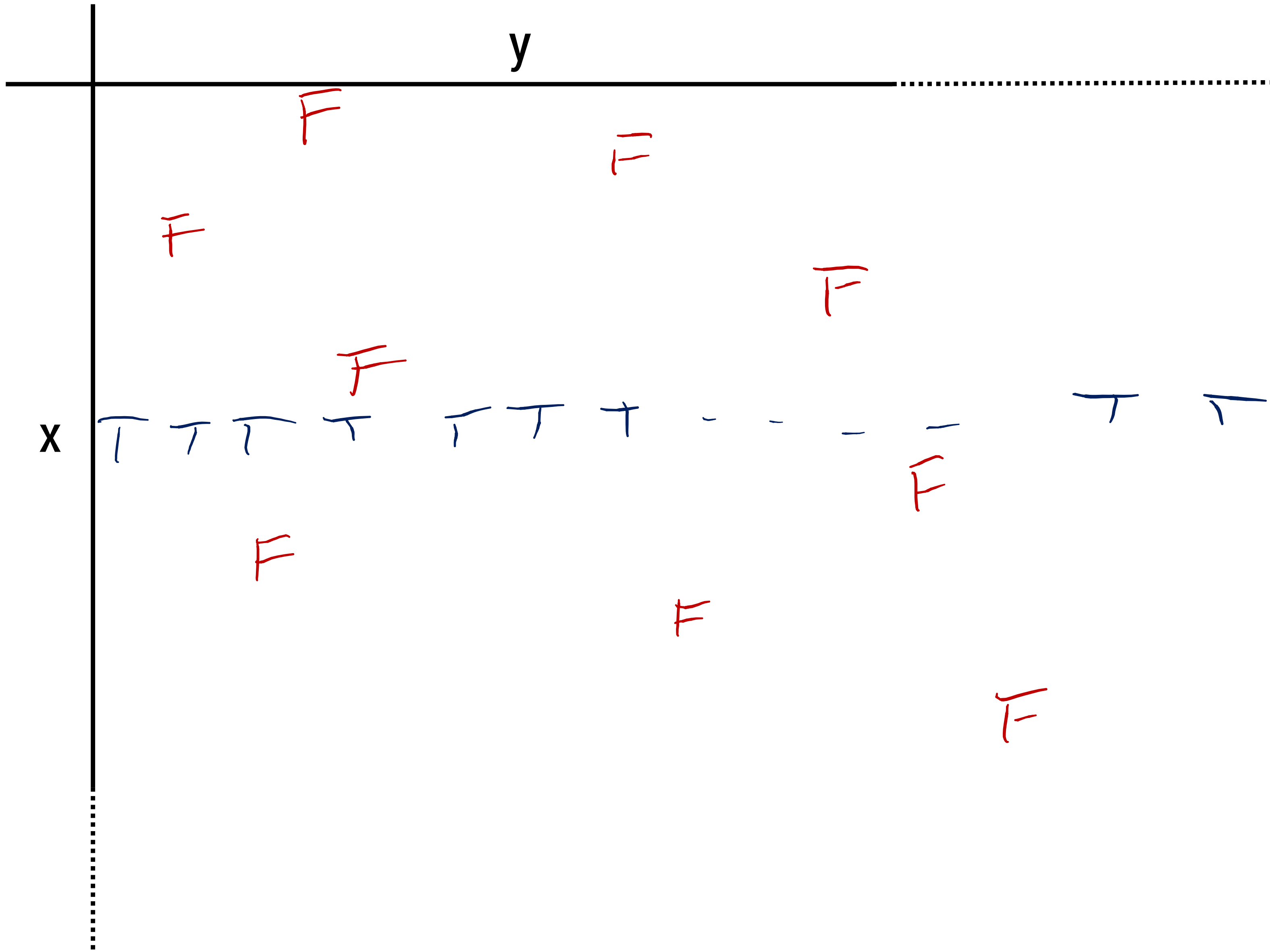
$\exists x\, \exists y\; P(x, y)$



$\forall x \, \exists y \, P(x, y)$



$\exists x \, \forall y \, P(x, y)$



quantification with two variables

expression	when **true**	when **false**
$\forall x \forall y P(x, y)$		
$\exists x \exists y P(x, y)$		
$\forall x \exists y P(x, y)$	(look	refine)
$\exists x \forall y P(x, y)$		

logal inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is only a small part of this

applications of logical inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

foundations of rational thought…

proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

an inference rule: *Modus Ponens*

- If p and p $\rightarrow$ q are both true then q must be true
- Write this rule as $$\frac{p,\ p \rightarrow q}{\therefore\ q}$$
- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.
- Therefore, by modus ponens:
  - You have a 311 class today.

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	given
2.	$p \rightarrow q$	given
3.	$q \rightarrow r$	given
4.	q	modus ponens from 1 and 2
5.	r	modus ponens from 3 and 4

proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	given
2.	$\neg q$	given
3.	$\neg q \rightarrow \neg p$	contrapositive of 1
4.	$\neg p$	modus ponens from 2 and 3

- Each inference rule is written as:

$$\frac{A,\ B}{\therefore\ C, D}$$

...which means that if both A and B are true then you can infer C and you can infer D.

  - For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies

- Sometimes rules don't need anything to start with. These rules are called axioms:
  - e.g. *Excluded Middle Axiom*

$$\frac{}{\therefore\ p \vee \neg p}$$

simple propositional inference rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it:

$$\frac{p \land q}{\therefore\ p,\ q}$$

$$\frac{p,\ q}{\therefore\ p \land q}$$

$$\frac{p \lor q\ ,\ \neg p}{\therefore\ q}$$

$$\frac{p}{\therefore\ p \lor q,\ q \lor p}$$

$$\frac{p,\ p \to q}{\therefore\ q}$$

$$\frac{p \Rightarrow q}{\therefore\ p \to q}$$

Direct Proof Rule

Not like other rules

important: applications of inference rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise)

e.g. ~~1. $p \rightarrow q$ given~~

~~2. $(p \vee r) \rightarrow q$ intro $\vee$ from 1.~~

Does not follow! e.g . p=**F**, q=**F**, r=**T**

direct proof of an implication

- $p \Rightarrow q$ denotes a proof of q given p as an assumption

- The direct proof rule:

  If you have such a proof then you can conclude

  that $p \rightarrow q$ is true

Example:

proof subroutine

1.	p	**assumption**
2.	$p \vee q$	intro for $\vee$ from 1
3.	$p \rightarrow (p \vee q)$	direct proof rule