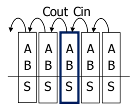
adding numbers in binary

$317 + 422 = (100111101)_2 + (110100110)_2$

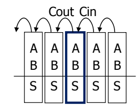
```
1 0 0 1 1 1 1 0 0
  100111101
  110100110
 1011100011
```



1-bit binary adder

- Inputs: A, B, Carry-in
- Outputs: Sum, Carry-out

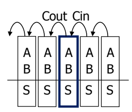
A	B	Cin	Cout	S
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

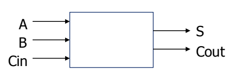


1-bit binary adder

- Inputs: A, B, Carry-in
- Outputs: Sum, Carry-out

A	B	Cin	Cout	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1





S = A′ B′ Cin + A′ B Cin′ + A B′ Cin′ + A B Cin

Cout = A′ B Cin + A B′ Cin + A B Cin′ + A B Cin

apply theorems to simplify expressions

The theorems of Boolean algebra can simplify expressions

– e.g., full adder's carry-out function

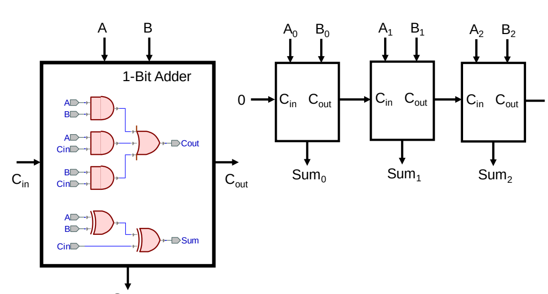
Cout = A′ B Cin + A B′ Cin + A B Cin′ + A B Cin
= A′ B Cin + A B′ Cin + A B Cin′ + A B Cin + A B Cin
= A′ B Cin + A B Cin + A B′ Cin + A B Cin′ + A B Cin
= (A′ + A) B Cin + A B′ Cin + A B Cin′ + A B Cin
= (1) B Cin + A B′ Cin + A B Cin′ + A B Cin
= B Cin + A B′ Cin + A B Cin′ + A B Cin + A B Cin
= B Cin + A B′ Cin + A B Cin + A B Cin′ + A B Cin
= B Cin + A (B′ + B) Cin + A B Cin′ + A B Cin
= B Cin + A (1) Cin + A B Cin′ + A B Cin
= B Cin + A Cin + A B (Cin′ + Cin)
= B Cin + A Cin + A B (1)
= B Cin + A Cin + A B

adding extra terms creates new factoring opportunities

cse 311: foundations of computing

Spring 2015
Lecture 5: Canonical forms and predicate logic

a 2-bit ripple-carry adder



mapping truth tables to logic gates

Given a truth table:

1. Write the Boolean expression
2. Minimize the Boolean expression
3. Draw as gates
4. Map to available gates

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

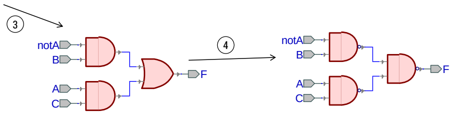
1

$F = A'BC' + A'BC + AB'C + ABC$

2

$= A'B(C' + C) + AC(B' + B)$

$= A'B + AC$



canonical forms

- Truth table is the unique signature of a Boolean function
- The same truth table can have many gate realizations
  - we've seen this already
  - depends on how good we are at Boolean simplification
- **Canonical forms**
  - standard forms for a Boolean expression
  - we all come up with the same expression

sum-of-products canonical form

- also known as Disjunctive Normal Form (DNF)
- also known as minterm expansion

F = *001* *011* *101* *110* *111*

$F = A'B'C + A'BC + AB'C + ABC' + ABC$

A	B	C	F	F′
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

sum-of-products canonical form

Product term (or minterm)

- ANDed product of literals – input combination for which output is true
- each variable appears exactly once, true or inverted (but not both)

A	B	C	minterms
0	0	0	A′B′C′
0	0	1	A′B′C
0	1	0	A′BC′
0	1	1	A′BC
1	0	0	AB′C′
1	0	1	AB′C
1	1	0	ABC′
1	1	1	ABC

F in canonical form:

$F(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC$

canonical form ≠ minimal form

$F(A, B, C) = A'B'C + A'BC + AB'C + ABC + ABC'$

$= (A'B' + A'B + AB' + AB)C + ABC'$

$= ((A' + A)(B' + B))C + ABC'$

$= C + ABC'$

$= ABC' + C$

$= AB + C$

product-of-sums canonical form

- Also known as Conjunctive Normal Form (CNF)
- Also known as maxterm expansion

F = *000* *010* *100*

$F = (A + B + C)\ (A + B' + C)\ (A' + B + C)$

A	B	C	F	F′
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

s-o-p, p-o-s, and de Morgan's theorem

Complement of function in sum-of-products form:

- $F' = A'B'C' + A'BC' + AB'C'$

Complement again and apply de Morgan's and get the product-of-sums form:

- $(F')' = (A'B'C' + A'BC' + AB'C')'$
- $F = (A + B + C)\ (A + B' + C)\ (A' + B + C)$

product-of-sums canonical form

Sum term (or maxterm)
- ORed sum of literals – input combination for which output is false
- each variable appears exactly once, true or inverted (but not both)

A	B	C	maxterms
0	0	0	A+B+C
0	0	1	A+B+C′
0	1	0	A+B′+C
0	1	1	A+B′+C′
1	0	0	A′+B+C
1	0	1	A′+B+C′
1	1	0	A′+B′+C
1	1	1	A′+B′+C′

F in canonical form:
F(A, B, C) = (A + B + C) (A + B′ + C) (A′ + B + C)

canonical form ≠ minimal form
F(A, B, C) = (A + B + C) (A + B′ + C) (A′ + B + C)
= (A + B + C) (A + B′ + C)
(A + B + C) (A′ + B + C)
= (A + C) (B + C)

predicate logic

- Propositional Logic
  - If Pikachu doesn't wear pants, then he flies on Bieber's jet unless Taylor is feeling lonely.
- Predicate Logic
  - If x, y, and z are positive integers, then $x^3 + y^3 \neq z^3$.

predicate logic

Predicate or Propositional Function
- A function that returns a truth value, e.g.,

"x is a cat"
"x is prime"
"student x has taken course y"
"x > y"
"x + y = z" or Sum(x, y, z)
"5 < x"

Predicates will have variables or constants as arguments.

domain of discourse

We must specify a "domain of discourse", which is the possible things we're talking about.

"x is a cat"
(e.g., mammals)

"x is prime"
(e.g., positive whole numbers)

student x has taken course y"
(e.g., students and courses)

quantifiers

$\forall x\, P(x)$
P(x) is true for every x in the domain
read as "for all x, P of x"

$\exists x\, P(x)$
There is an x in the domain for which P(x) is true
read as "there exists x, P of x"

statements with quantifiers

Domain:
Positive Integers

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
(or "x>y")
Equal(x,y)
(or "x=y")
Sum(x,y,z)

- $\exists$x Even(x)
- $\forall$x Odd(x)
- $\forall$x (Even(x) $\vee$ Odd(x))
- $\exists$x (Even(x) $\wedge$ Odd(x))
- $\forall$x Greater(x+1, x)
- $\exists$x (Even(x) $\wedge$ Prime(x))

statements with quantifiers

- $\forall x\ \exists y$ Greater (y, x)
- $\forall x\ \exists y$ Greater (x, y)
- $\forall x\ \exists y$ (Greater(y, x) $\wedge$ Prime(y))
- $\forall x$ (Prime(x) $\rightarrow$ (Equal(x, 2) $\vee$ Odd(x))
- $\exists x\ \exists y$ (Sum(x, 2, y) $\wedge$ Prime(x) $\wedge$ Prime(y))

Domain:
Positive Integers

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
(or "x>y")
Equal(x,y)
(or "x=y")
Sum(x,y,z)

statements with quantifiers

- $\forall x\ \exists y$ Greater (y, x) T
- $\forall x\ \exists y$ Greater (x, y) F

Domain:
All integers

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
(or "x>y")
Equal(x,y)
(or "x=y")
Sum(x,y,z)

Domain of quantifiers is important!

English to predicate logic

- "Red cats like tofu"
- "Some red cats don't like tofu"

Cat(x)
Red(x)
LikesTofu(x)

negations of quantifiers

- not every positive integer is prime
- some positive integer is not prime
- prime numbers do not exist
- every positive integer is not prime

negations of quantifiers

- $\forall x$ PurpleFruit(x)
  - "All fruits are purple"
  - What is $\neg\forall x$ PurpleFruit(x)
  - "Not all fruits are purple"
- How about $\exists x$ PurpleFruit(x)?
  - "There is a purple fruit"
  - If it's the negation, all situations should be covered by a statement and its negation.
  - Consider the domain {Orange}: Neither statement is true!
  - No.
- How about $\exists x\ \neg$PurpleFruit(x)?
  - "There is a fruit that isn't purple"
  - Yes.

Domain:
Fruit

PurpleFruit(x)

de Morgan's laws for quantifiers

$$\boxed{\begin{array}{l} \neg\forall x\ P(x) \equiv \exists x\ \neg P(x) \\ \neg\exists x\ P(x) \equiv \forall x\ \neg P(x) \end{array}}$$

de Morgan's laws for quantifiers

$$\boxed{\begin{array}{l} \neg\forall x\ P(x) \equiv \exists x\, \neg P(x) \\ \neg\exists x\ P(x) \equiv \forall x\, \neg P(x) \end{array}}$$

"There is no largest integer."

$$\begin{aligned} & \neg\exists x\ \forall y\ (x \geq y) \\ \equiv\ & \forall x\, \neg\forall y\ (x \geq y) \\ \equiv\ & \forall x\ \exists y\, \neg(x \geq y) \\ \equiv\ & \forall x\ \exists y\ (y > x) \end{aligned}$$

"For every integer there is a larger integer."

scope of quantifiers

example: $\text{Notlargest}(x) \equiv \exists y\ \text{Greater}(y, x)$
$\equiv \exists z\ \text{Greater}(z, x)$

truth value:

doesn't depend on y or z "bound variables"

does depend on x "free variable"

quantifiers only act on free variables of the formula they quantify

$\forall x\ (\exists y\ (P(x, y) \to \forall x\ Q(y, x)))$

scope of quantifiers

$\exists x\ (P(x) \land Q(x))$ vs. $\exists x\ P(x) \land \exists x\ Q(x)$