Spring 2015

Lecture 19: Structural induction and regular expressions

- An *alphabet* **Σ** is any finite set of characters.

  e.g. $\Sigma = \{0,1\}$ or $\Sigma = \{A, B, C, \ldots X, Y, Z\}$ or

  $\Sigma =$

1		28	∟	95	_	153	Ö	186	║	219	█
2	☻	29	↔	96	`	154	Ü	187	╗	220	▄
3	♥	30	▲	97-122	a-z	155	¢	188	╝	221	▌
4	♦	31	▼	123	{	156	£	189	╜	222	▐
5	♣	32	(space)	124	|	157	¥	190	╛	223	▀
6	♠	33	!	125	}	158	₧	191	┐	224	α
7	•	34	"	126	~	159	ƒ	192	└	225	ß
8	◘	35	#	127	⌂	160	á	193	┴	226	Γ
9	○	36	$	128	Ç	161	í	194	┬	227	π
10	◙	37	%	129	ü	162	ó	195	├	228	Σ
11	♂	38	&	130	é	163	ú	196	─	229	σ

- The set **Σ*** of *strings* over the alphabet **Σ** is defined by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Length:

len (ε) = 0;

len (wa) = 1 + len(w); for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^{\mathrm{R}} = \varepsilon$

$(wa)^{\mathrm{R}} = aw^{\mathrm{R}}$ for $w \in \Sigma^*$, $a \in \Sigma$

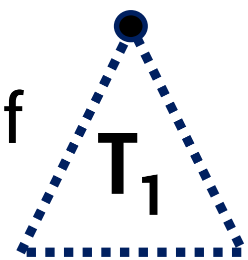
Concatenation:

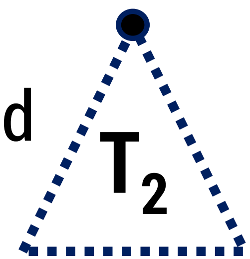
$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x, w \in \Sigma^*$, $a \in \Sigma$

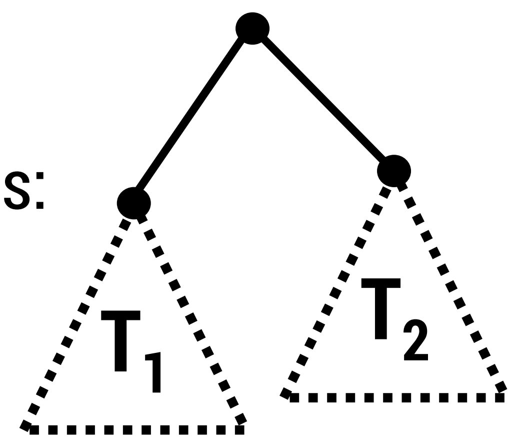
rooted binary trees

- Basis: • is a rooted binary tree
- Recursive step:

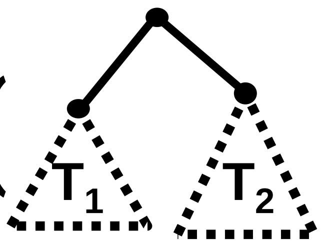
If 

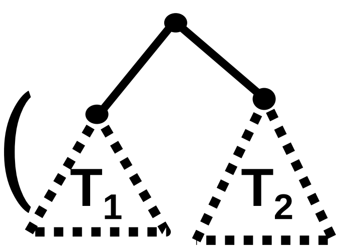
and 

are rooted binary trees,

then so is: 

defining a function on rooted binary trees

- $\text{size}(\bullet) = 1$
- size (

) $= 1 + \text{size}(T_1) + \text{size}(T_2)$
- $\text{height}(\bullet) = 0$
- height (

) $=1 + \max\{\text{height}(T_1), \text{height}(T_2)\}$

structural induction

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

structural induction for strings

Let S be a set of strings over $\Sigma = \{a, b\}$ defined by

Basis: $a \in S$

Recursive:

If $w \in S$ then $wa \in S$ and $wba \in S$

If $u, v \in S$ then $uv \in S$

Claim: If $w \in S$ then w has more a's than b's.

proof continued?

prove: $\mathrm{len}(x \cdot y) = \mathrm{len}(x) + \mathrm{len}(y)$ for all $x, y \in \Sigma^*$

Let $P(y)$ be "$\mathrm{len}(x \cdot y) = \mathrm{len}(x) + \mathrm{len}(\mathrm{y})$ for all $x \in \Sigma^*$

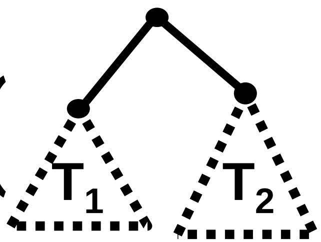
Length:

len (ε) = 0;

len (wa) = 1 + len(w); for $w \in \Sigma^*, a \in \Sigma$

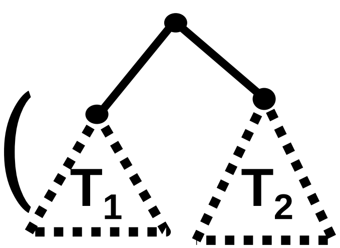
defining a function on rooted binary trees

- size(•) = 1

- size (

) = 1 + size(T_1) + size(T_2)

- height(•) = 0

- height (

) =1 + max{height(T_1), height(T_2)}

Claim: For every rooted binary tree T, $\mathrm{size}(T) \leq 2^{\mathrm{height}(T)+1} - 1$

languages: sets of strings

Sets of strings that satisfy special properties are called languages.

Examples:

- English sentences
- Syntactically correct Java/C/C++ programs
- $\Sigma^* =$ All strings over alphabet Σ
- Palindromes over Σ
- Binary strings that don't have a 0 after a 1
- Legal variable names, keywords in Java/C/C++
- Binary strings with an equal # of 0's and 1's