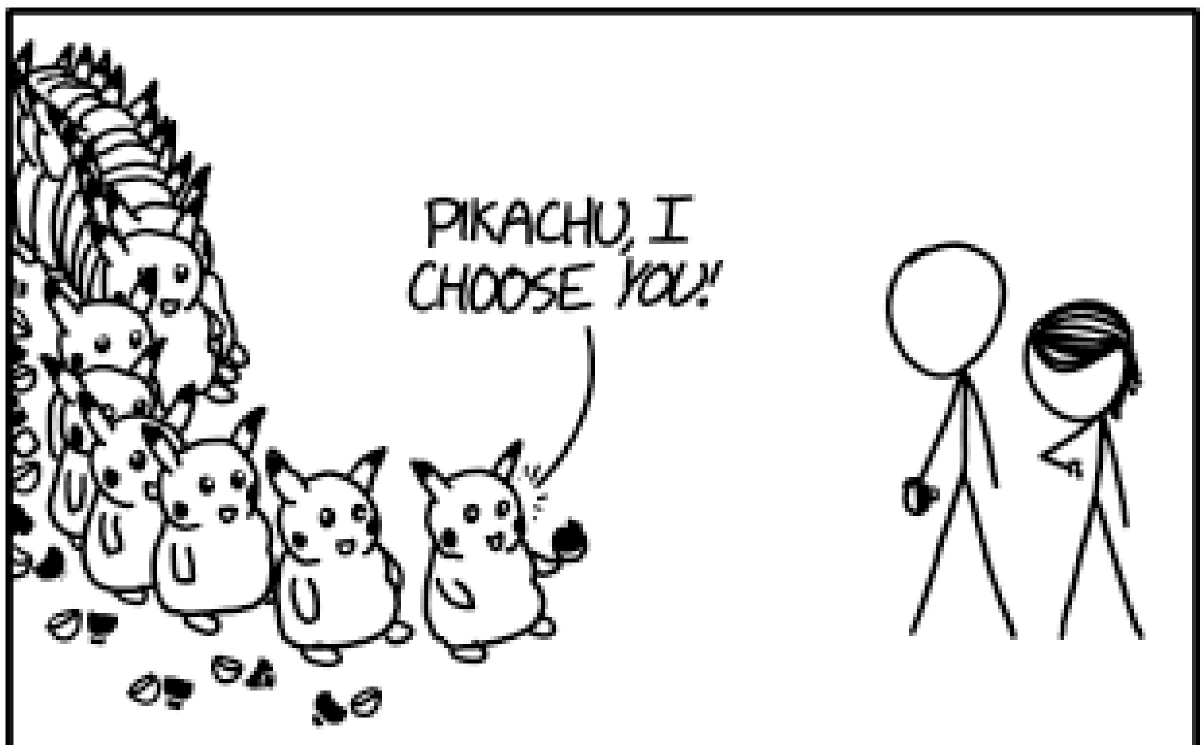
Fall 2015

Lecture 18:
Recursively defined sets and structural induction



Four weeks left: What happens now?

The class speeds up a bit.
Homework problems get more conceptual.

We will cover:

- Recursively defined sets and functions
- Structural induction
- Regular expressions and context free grammars
- Relations and graphs
- Finite state machines and automata
- Turing machines and undecidability

Will give exams back at the end of class.

Regrade requests: Each problem belongs to a TA

Problem 1	Robbie
Problem 2	Becky / Tim
Problem 3	Jiechen
Problem 4	Sam
Problem 5	Ian
Problem 6	Evan

You have to go see the relevant grader.

As usual:

Make sure you understand the problem and your solution.

stressed?

Recursive definition

- Basis step: $0 \in S$
- Recursive step: if $x \in S$, then $x + 2 \in S$
- Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

recursive definition of sets

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$, then $x + y \in S$;

Basis: $[1, 1, 0] \in S, [0, 1, 1] \in S$;

Recursive:

if $[x, y, z] \in S$, $\alpha \in \mathbb{R}$, then $[\alpha x, \alpha y, \alpha z] \in S$

if $[x_1, y_1, z_1], [x_2, y_2, z_2] \in S$

then $[x_1 + x_2, \; y_1 + y_2, \; z_1 + z_2] \in S$

Powers of 3:

recursive definitions of sets: general form

Recursive definition

– *Basis step:* Some specific elements are in S
– *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
– *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

- An *alphabet* **Σ** is any finite set of characters.

  e.g. $\Sigma = \{0,1\}$ or $\Sigma = \{A, B, C, \dots X, Y, Z\}$ or

  $\Sigma =$

1		28	∟	95	_	153	Ö	186	║	219	█
2	☻	29	↔	96	`	154	Ü	187	╗	220	▄
3	♥	30	▲	97-122	a-z	155	¢	188	╝	221	▌
4	♦	31	▼	123	{	156	£	189	╜	222	▐
5	♣	32	(space)	124	|	157	¥	190	╛	223	▀
6	♠	33	!	125	}	158	₧	191	┐	224	α
7	•	34	"	126	~	159	ƒ	192	└	225	ß
8	◘	35	#	127	⌂	160	á	193	┴	226	Γ
9	○	36	$	128	Ç	161	í	194	┬	227	π
10	◙	37	%	129	ü	162	ó	195	├	228	Σ
11	♂	38	&	130	é	163	ú	196	─	229	σ

- The set **Σ*** of *strings* over the alphabet **Σ** is defined by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes are strings that are the same backwards and forwards.

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$.

First digit cannot be a 1.

* No occurrence of the substring 11.

function definitions on recursively defined sets

Length:

len $(\varepsilon) = 0$;

len $(wa) = 1 + \text{len}(w)$; for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^{\mathrm{R}} = \varepsilon$

$(wa)^{\mathrm{R}} = aw^{\mathrm{R}}$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x, w \in \Sigma^*$, $a \in \Sigma$

function definitions on recursively defined sets

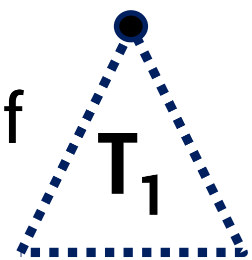
Number of vowels in a string:

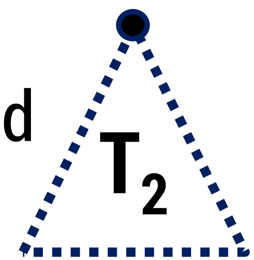
$$\Sigma = \{a, b, c, \dots, z\}$$

$$\mathcal{V} = \{a, e, i, o, u\}$$

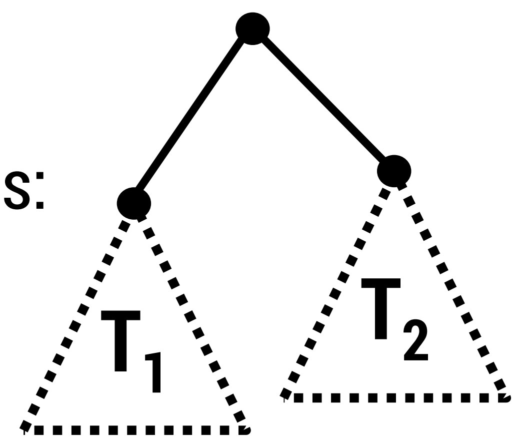
rooted binary trees

- Basis: • is a rooted binary tree
- Recursive step:

If 

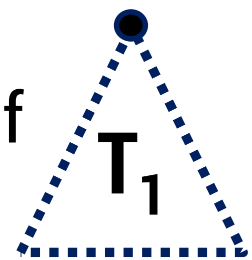
and 

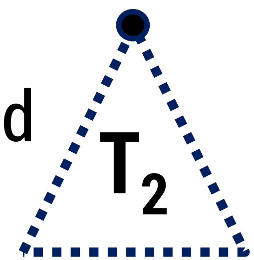
are rooted binary trees,

then so is: 

rooted binary trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

and 

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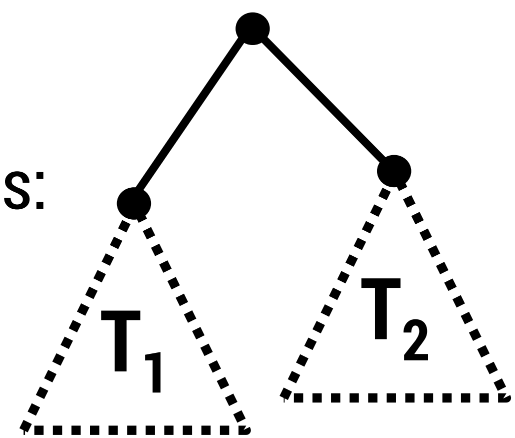
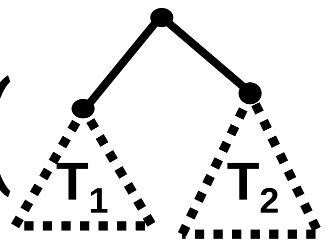
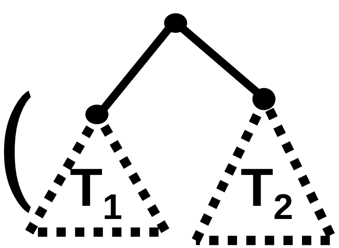
then so is: 

defining a function on rooted binary trees

- size(•) = 1
- size (

) = 1 + size(T_1) + size(T_2)
- height(•) = 0
- height (

)=1 + max{height(T_1), height(T_2)}

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

How to prove $\forall\, x \in S,\, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

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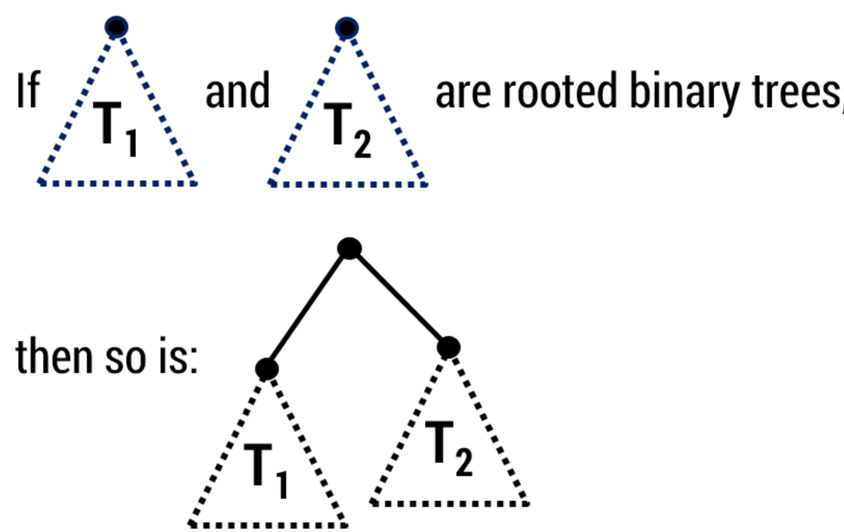
Conclude that $\forall\, x$

- Basis: • is a rooted binary tree
- Recursive step:



structural induction vs. ordinary induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Let $Q(n)$ be true iff for all $x \in S$ that take n recursive steps to be constructed, $P(x)$ is true.

using structural induction

Let S be given by:

- **Basis:** $6 \in S$; $15 \in S$;
- **Recursive:** if $x, y \in S$ **then** $x + y \in S$.

Claim: Every element of S is divisible by 3.