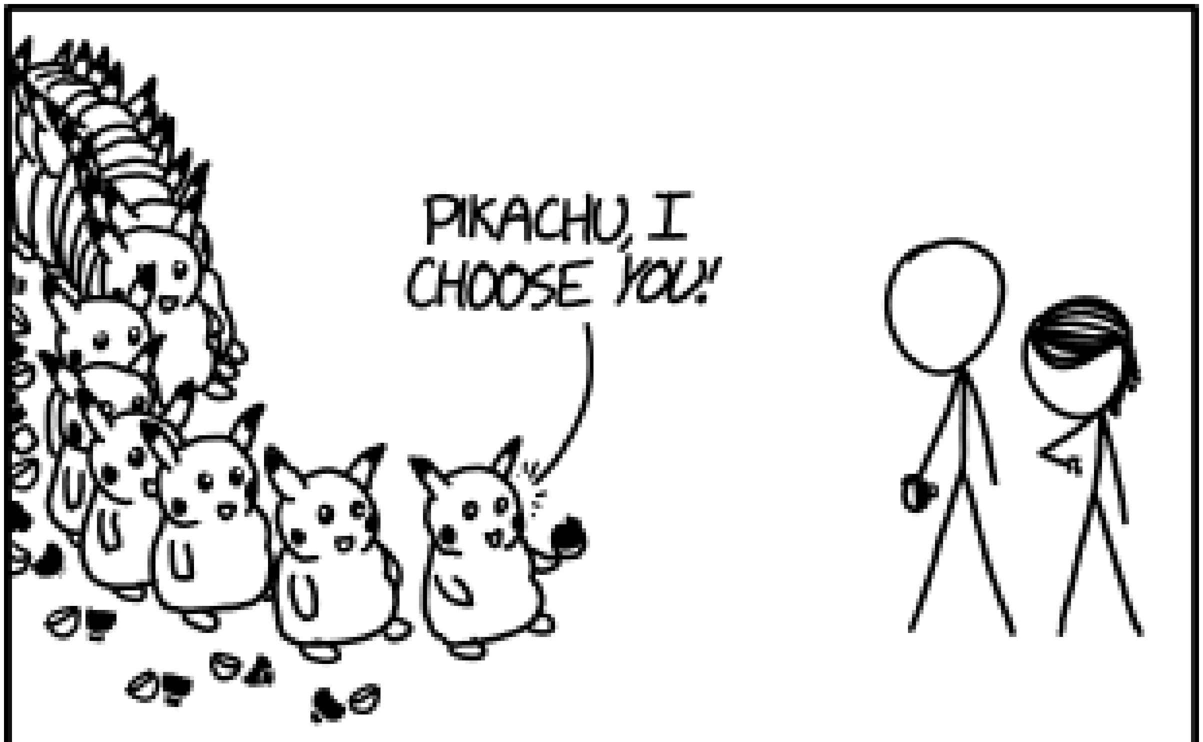
Fall 2015

Lecture 18:
Recursively defined sets and structural induction



Four weeks left: What happens now?

The class speeds up a bit.
Homework problems get more conceptual.

We will cover:

- Recursively defined sets and functions
- Structural induction
- Regular expressions and context free grammars
- Relations and graphs
- Finite state machines and automata
- Turing machines and undecidability

Will give exams back at the end of class.

Regrade requests: Each problem belongs to a TA

Problem 1	Robbie
Problem 2	Becky / Tim
Problem 3	Jiechen
Problem 4	Sam
Problem 5	Ian
Problem 6	Evan

You have to go see the relevant grader.

As usual:
Make sure you understand the problem and your solution.

stressed?

Recursive definition

- Basis step: $0 \in S$
- Recursive step: if $x \in S$, then $x + 2 \in S$
- Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

S = positive even numbers

$= \{0, 2, 4, \ldots\}$

$3^0 = 1$, $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ recursive definition of sets

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$, then $x + y \in S$;

$S = \{ 6a + 15b : a, b$ non-neg integers $\}$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$;

Recursive:

if $[x, y, z] \in S$, $\alpha \in \mathbb{R}$, then $[\alpha x, \alpha y, \alpha z] \in S$

if $[x_1, y_1, z_1], [x_2, y_2, z_2] \in S$

then $[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$

$\text{span}\{[1,1,0], [0,1,1]\} \subseteq \mathbb{R}^3$

Powers of 3:

Basis: $1 \in S$

maybe $x \in S \Rightarrow \frac{x}{3} \in S$

Recursive: $x \in S \Rightarrow 3x \in S$

recursive definitions of sets: general form

Recursive definition

– *Basis step:* Some specific elements are in S

– *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.

– *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

$w, w' \in \Sigma^*$

- An *alphabet* **Σ** is any finite set of characters.

  e.g. $\Sigma = \{0,1\}$ or $\Sigma = \{A, B, C, \ldots X, Y, Z\}$ or

  $\Sigma =$

1		28	∟	95	_	153	Ö	186	║	219	█
2	☻	29	↔	96	`	154	Ü	187	╗	220	▄
3	♥	30	▲	97-122	a-z	155	¢	188	╝	221	▌
4	♦	31	▼	123	{	156	£	189	╜	222	▐
5	♣	32	(space)	124	|	157	¥	190	╛	223	▀
6	♠	33	!	125	}	158	₧	191	┐	224	α
7	●	34	"	126	~	159	ƒ	192	└	225	ß
8	◘	35	#	127	⌂	160	á	193	┴	226	Γ
9	○	36	$	128	Ç	161	í	194	┬	227	π
10	◙	37	%	129	ü	162	ó	195	├	228	Σ
11	♂	38	&	130	é	163	ú	196	─	229	σ

Check Wikipedia

- The set **Σ*** of *strings* over the alphabet **Σ** is defined by
  - **Basis:** $\mathcal{E} \in \Sigma^*$ ($\mathcal{E}$ is the empty string) (λ also used)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

$\Sigma = \{0,1\}$ $\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \ldots\}$

Palindromes are strings that are the same backwards and forwards.

Σ Some finite alphabet

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$.

$\Sigma = \{0, 1\}$

$x \in S \Longrightarrow$ if $(y \neq \varepsilon)$ $y1 \in S$

$y \in S$

$z \in S$

binary strings such that…

First digit cannot be a 1.

Basis: $\varepsilon \in S$

Recursive: $x \in S \Longrightarrow x0 \in S$

✓ (OK) $y \in S$ and $y \neq \varepsilon$ then $y1 \in S$

Basis: $0 \in S$ *

Recursive: $x \in S \Rightarrow 0x \in S$

$x \in S \Rightarrow x1 \in S$

$x \in S \Rightarrow \varepsilon \in S$

* No occurrence of the substring 11.

Basis: $\varepsilon \in S$, $1 \in S$

Rec: $x \in S \Rightarrow x0 \in S$

$x \in S \Rightarrow x01 \in S$

✓

$\leftarrow$ prove it works?

function definitions on recursively defined sets

Length:

len $(\varepsilon) = 0$;

len $(wa) = 1 +$ len(w); for $w \in \Sigma^*, a \in \Sigma$

Reversal:

$(011)^R = 1(01)^R$

$= 110^R = 110\,\varepsilon^R$

$= 110.$

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*, a \in \Sigma$

$\boxed{(xa)^R = ax^R}$

$(x \cdot y)^R = y^R \cdot x^R$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x, w \in \Sigma^*, a \in \Sigma$

$x \cdot (011) = (x \cdot 01)1 = (x \cdot 0)11$

$= (x \cdot \varepsilon)011 = x011.$

function definitions on recursively defined sets

Number of vowels in a string:

$\Sigma = \{a, b, c, \ldots, z\}$

$\mathcal{V} = \{a, e, i, o, u\}$

$w \in \Sigma^*$

$wa \in \Sigma^*$

for $a \in \Sigma$

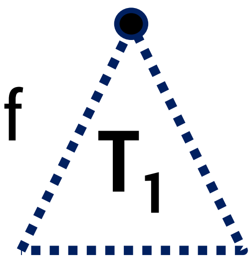
$NV(x) =$ # vowels in x

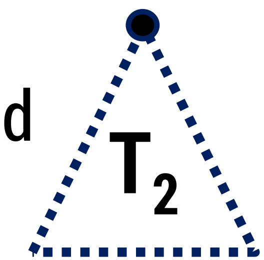
$NV(\varepsilon) = 0$

$$NV(wa) = \begin{cases} \text{if } a \in \mathcal{V} & 1 + NV(w) \\ \text{if } a \notin \mathcal{V} & NV(w) \end{cases}$$

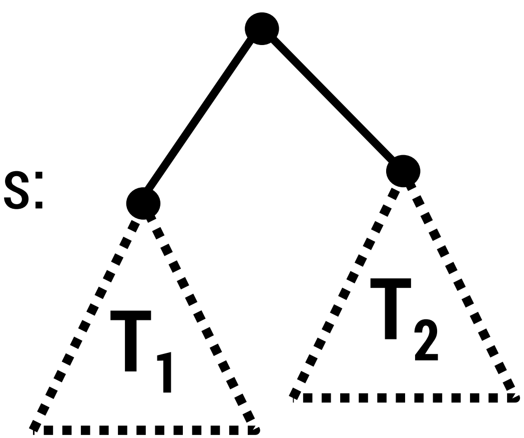
rooted binary trees

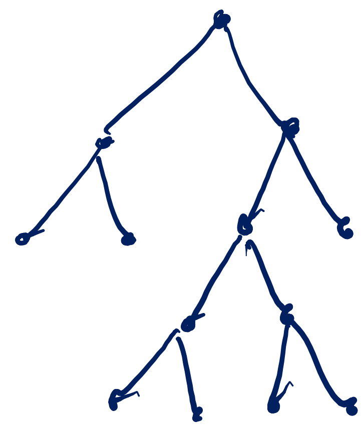
- Basis: • is a rooted binary tree
- Recursive step:

If 

and 

are rooted binary trees,

then so is: 



rooted binary trees

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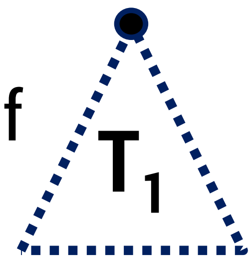
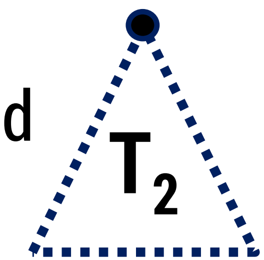
If 

and 

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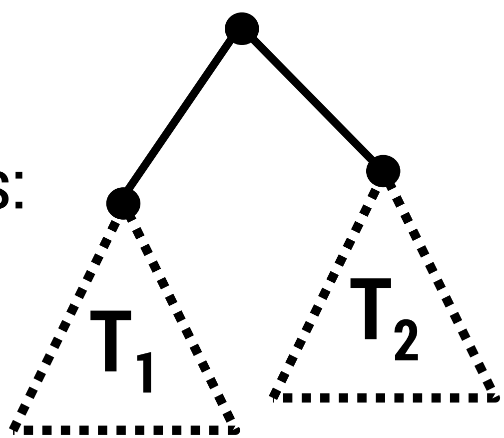
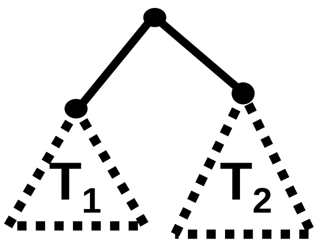
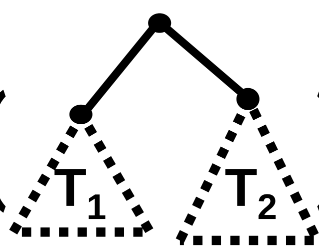
then so is: 

defining a function on rooted binary trees

- size(•) = 1
- size () = 1 + size(T_1) + size(T_2)

- height(•) = 0
- height () = 1 + max{height(T_1), height(T_2)}

How to prove $\forall\ x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\ x \in S, P(x)$

How to prove $\forall\, x \in S,\, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothe
arbitrary values of
mentioned in the *R*

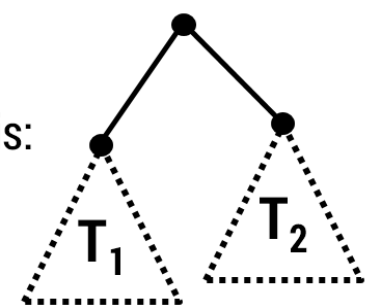
Inductive Step: Pr
elements w const
named elements m

Conclude that $\forall\, x$

- Basis: • is a rooted binary tree
- Recursive step:

If T_1 and T_2 are rooted binary trees,

then so is:



structural induction vs. ordinary induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Let $Q(n)$ be true iff for all $x \in S$ that take n recursive steps to be constructed, $P(x)$ is true.

Let S be given by:

- **Basis:** $6 \in S$; $15 \in S$;
- **Recursive:** if $x, y \in S$ **then** $x + y \in S$.

Claim: Every element of S is divisible by 3.