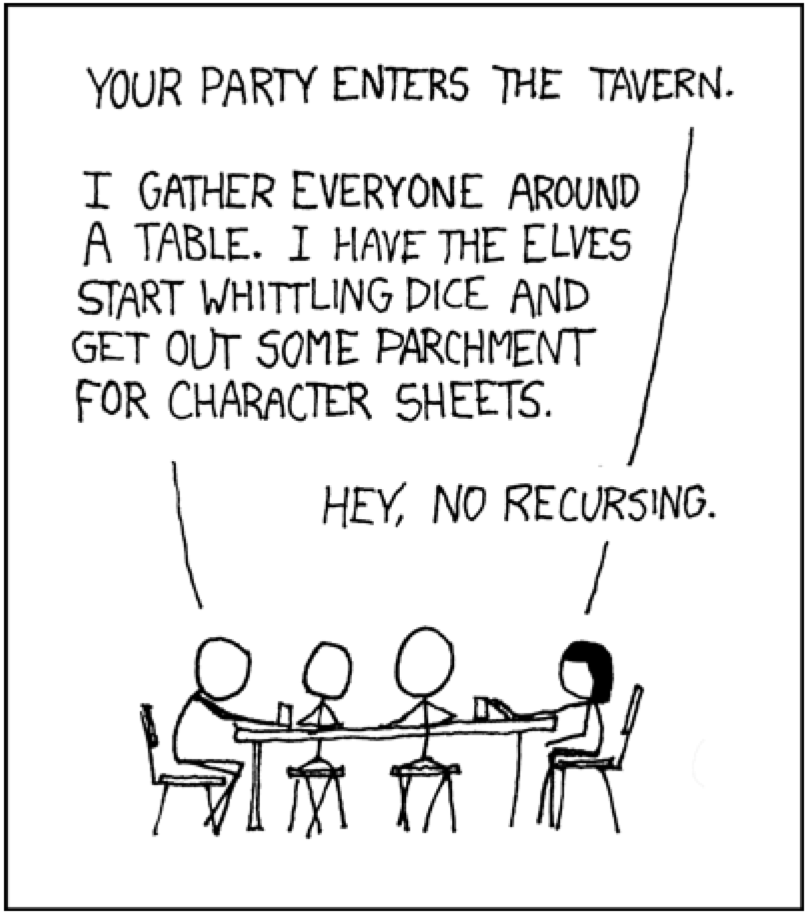
Fall 2015

Lecture 17: Strong induction & Recursive definitions



administrative

Midterm review session Sunday @ 1:00 pm (EEB 105)

MIDTERM MONDAY (IN THIS ROOM, USUAL TIME)

No office hours on Monday/Wednesday

Closed book.
One page (front and back) of notes allowed.

Exam includes induction!
Homework #5 is due on Friday, Nov 13th.

review: strong induction

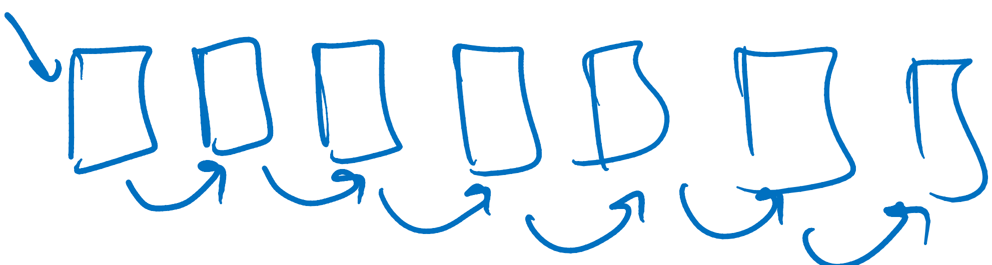
$$P(0)$$

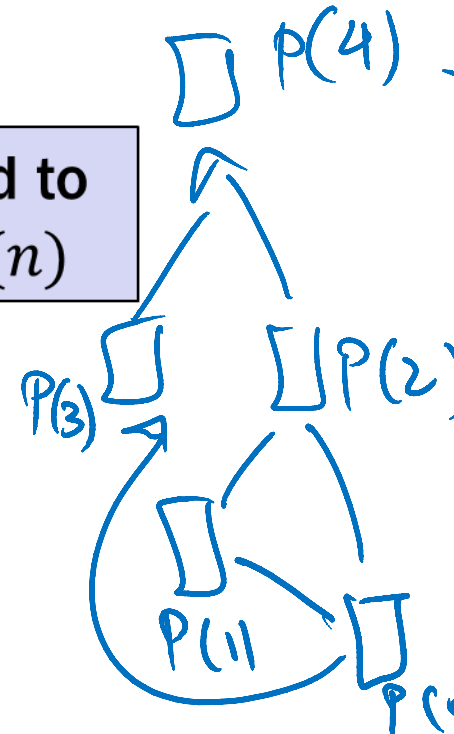
$$\forall k \ \Big(\big(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k) \big) \rightarrow P(k+1) \Big)$$

$$\therefore \forall n \ P(n)$$

Follows from ordinary induction applied to

$$Q(n) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(n)$$





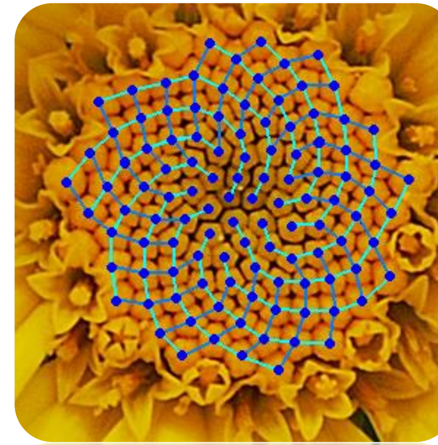
review: strong induction English proof

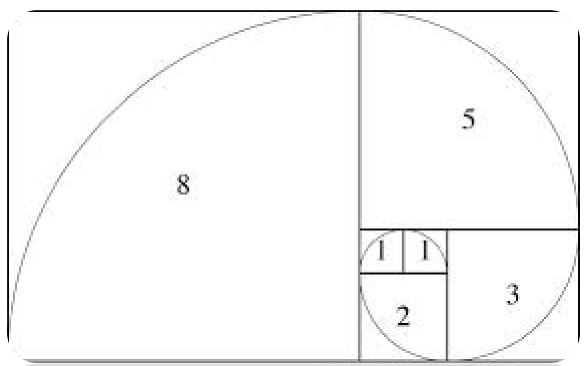
1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

review: Fibonacci numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

$f_2 = 1$
$f_3 = 2$





Theorem: $f_n < 2^n$ for all $n \geq 2$.

$P(n) =$ " $f_n < 2^n$ ".

Base Case $P(2)$ $f_2 = 1 < 2^2 = 4$

$P(3)$ $f_3 = 2 < 2^3 = 8$

IH: For some $k \geq$ ~~2~~ 3 and any $2 \leq j \leq k$, $P(j)$ holds

IS: Goal $P(k+1)$ holds.

$f_{k+1} = f_k + f_{k-1}$ since $k-1 \geq 2$, $P(k), P(k-1)$ holds

$f_k < 2^k$, $f_{k-1} < 2^{k-1}$

$f_{k+1} < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$.

$P(k+1)$ holds.

bounding the Fibonacci numbers

Theorem: $2^{\frac{n}{2}-1} \leq f_n < 2^n$ for all $n \geq 2$

$P(n) = $ " $2^{n/2-1} \leq f_n < 2^n$ ".

Base Case $\quad 2^0 \leq 1 = f_2 < 2^2 \qquad P(2)$ ✓

$\sqrt{2} \leq 2 = f_3 \leq 2^3 \qquad P(3)$

IH: For some $k \geq 3$, and any $2 \leq j \leq k$, $P(j)$ holds.

IS: Goal $P(k+1)$ holds

$f_{k+1} = f_k + f_{k-1}$

$< 2^k + 2^{k-1}$

$< 2^{k+1}$

$f_{k+1} \overset{?}{\geq} 2^{\frac{k+1}{2}-1} = 2^{\frac{k+2-1}{2}-1} = 2^{\frac{k-1}{2}}$

$f_{k+1} = f_k + f_{k-1}$

$\geq 2^{\frac{k}{2}-1} + 2^{\frac{k-1}{2}-1}$

$\geq 2^{\frac{k-1}{2}-1} + 2^{\frac{k-1}{2}-1}$

$= 2 \cdot 2^{\frac{k-1}{2}-1} = 2^{\frac{k+1}{2}-1}$

$f_0 = 0$; $f_1 = 1$; $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Theorem: $2^{n/2-1} \leq f_n < 2^n$ **for all** $n \geq 2$

Proof:

1. Let P(n) be "$2^{n/2-1} \leq f_n < 2^n$. By (strong) induction we prove P(n) for all $n \geq 2$.
2. **Base Case:** P(2) is true: $f_2=1$, $2^{2/2-1}=2^0=1 \leq f_2$, $2^2=4>f_2$
3. **Ind.Hyp:** Assume $2^{j/2-1} \leq f_j < 2^j$ for all integers j with $2 \leq j \leq k$ for for some arbitrary integer $k \geq 2$.
4. **Ind. Step:** Goal: Show $2^{(k+1)/2-1} \leq f_{k+1} < 2^{k+1}$

Case k=2: P(3) is true: $f_3=f_2+f_1=1+1=2$, $2^{3/2-1}=2^{1/2} \leq 2 = f_3$, $2^3=8 > f_3$

Case k≥3:

$$f_{k+1} = f_k + f_{k-1} \geq 2^{k/2-1} + 2^{(k-1)/2-1} \quad \text{by I.H. since } k-1 \geq 2$$

$$> 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2 \cdot 2^{(k-1)/2-1} = 2^{(k+1)/2-1}$$

$$f_{k+1} = f_k + f_{k-1} < 2^k + 2^{(k-1)} \quad \text{by I.H. since } k-1 \geq 2$$

$$< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Theorem: For any positive integers n, d, there are integers q, r such that $n = dq + r$ and $0 \leq r \leq d - 1$.

$n \geq 0$

Choose arbitrary $d \geq 1$

$P(n) =$ "$\exists q, r$ $n = dq + r$ and $0 \leq r \leq d-1$". $n \geq 0$

Base Case $P(0)$ holds $0 = 0 \cdot d + 0$ $0 \leq 0 \leq d-1$

$P(n)$ $\forall$ $n < d$. $n = 0 \cdot d + n$ $0 \leq n \leq d-1$ ✓

IH: For some $k \geq d-1$ and any $0 \leq j \leq k$, $P(j)$ holds

IS: Goal $P(k+1)$ holds

$P(k+1-d)$ holds. because $0 \leq k+1-d \leq k$

$\exists q, r$ s.t. $k+1-d = q \cdot d + r$ and $0 \leq r \leq d-1$

$P(k+1)$ holds $k+1 = (1+q)d + r$,

running time of Euclid's algorithm

$a > b$

$\gcd(a, b) = \gcd(b, a \bmod b)$

$\gcd(b, 0) = 0$

running time of Euclid's algorithm

$\forall a, b \ \exists s, t$ $sa + tb = \gcd(a,b)$

Theorem: Suppose that Euclid's algorithm takes n steps for $\gcd(a, b)$ with $a > b$, then $a \geq f_{n+1}$. $> 2^{\frac{n+1}{2}-1}$

gcd takes $O(\lg n)$ steps.

Proof:

Set $r_{n+1} = a, r_n = b$ then Euclid's algorithm computes

$$r_{n+1} = q_n r_n + r_{n-1} \quad (q_n \geq 1)$$
$$r_n = q_{n-1} r_{n-1} + r_{n-2}$$
$$\vdots$$
$$r_3 = q_2 r_2 + r_1$$
$$r_2 = q_1 r_1$$

r_1

Prove Indn

each quotient $q_i \geq 1$

$r_1 \geq 1$

$r_1 \geq f_2 = 1$

$r_2 \geq 2 = f_3$.

$r_{n+1} \geq r_n + r_{n-1}$

$\geq f_n + f_{n-1}$

$= f_{n+1}$

Recursive definition

- Basis step: $0 \in S$
- Recursive step: if $x \in S$, then $x + 2 \in S$
- Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

Basis: $6 \in S$; $15 \in S$;
Recursive: if $x, y \in S$, then $x + y \in S$;

Basis: $[1, 1, 0] \in S, [0, 1, 1] \in S$;
Recursive:
if $[x, y, z] \in S$, $\alpha \in \mathbb{R}$, then $[\alpha x, \alpha y, \alpha z] \in S$
if $[x_1, y_1, z_1], [x_2, y_2, z_2] \in S$
then $[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$

Powers of 3:

recursive definitions of sets: general form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

- An *alphabet* Σ is any finite set of characters.

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes are strings that are the same backwards and forwards.

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$.

all binary strings with no 1’s before 0’s

function definitions on recursively defined sets

Length:

len $(\varepsilon) = 0$;

len $(wa) = 1 +$ len(w); for $w \in \Sigma^*, a \in \Sigma$

Reversal:

$\varepsilon^{\mathrm{R}} = \varepsilon$

$(wa)^{\mathrm{R}} = aw^{\mathrm{R}}$ for $w \in \Sigma^*, a \in \Sigma$

Concatenation:

function definitions on recursively defined sets

Length:

len (ε) = 0;

len (wa) = 1 + len(w); for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^{\mathrm{R}} = \varepsilon$

$(wa)^{\mathrm{R}} = aw^{\mathrm{R}}$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x, w \in \Sigma^*$, $a \in \Sigma$