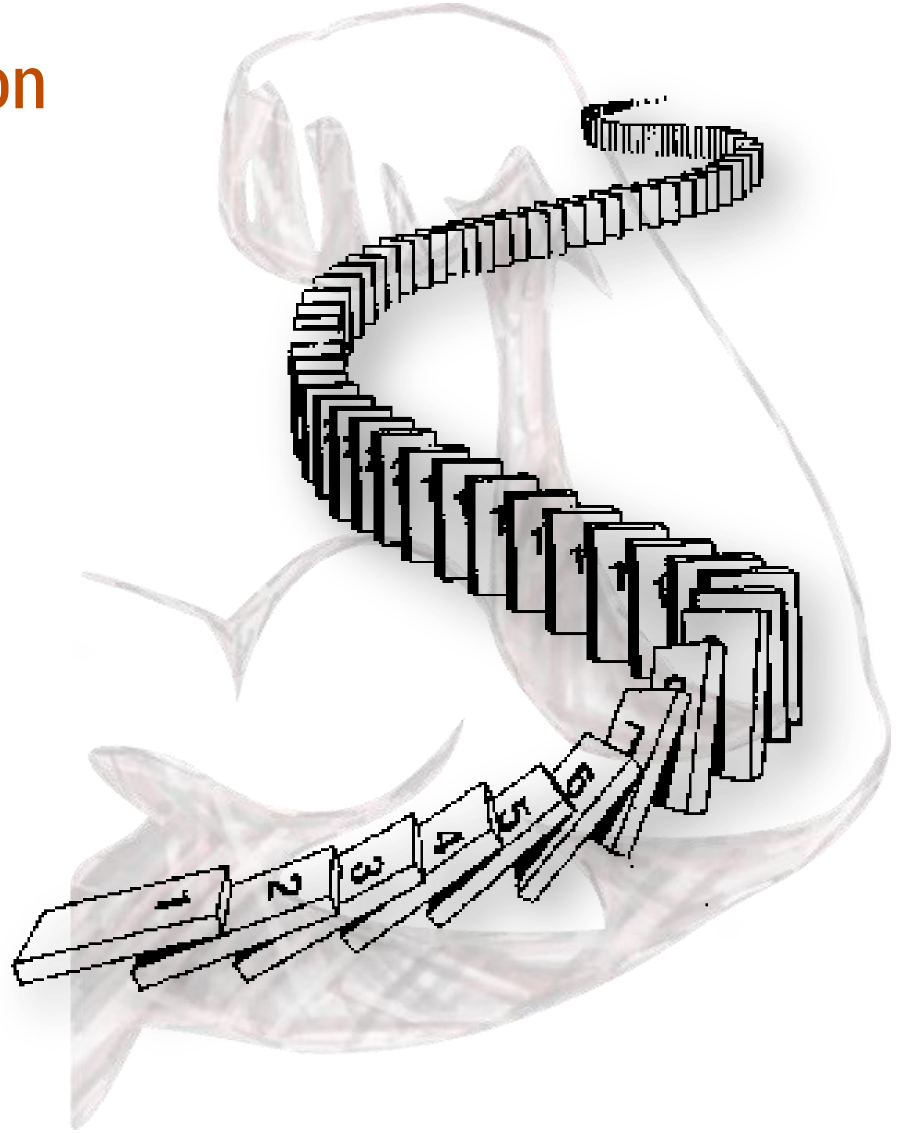
Spring 2015

Lecture 16: **Strong** induction



review: **induction** is a rule of inference

Domain: Natural Numbers

$$
\begin{array}{l}
P(0) \\
\forall\, k\, (P(k) \;\rightarrow\; P(k+1)) \\
\hline
\therefore\, \forall\, n\, P(n)
\end{array}
$$

review: using the induction rule in a formal proof

$$\begin{array}{l} P(0) \\ \forall\, k\ (P(k)\ \rightarrow\ P(k+1)) \\ \hline \end{array}$$

$$\therefore\ \forall\, n\ P(n)$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

review: format of an induction proof

$$P(0)$$
$$\forall\, k\ (P(k)\ \rightarrow\ P(k+1))$$
$$\therefore\ \forall\, n\ P(n)$$

1. Prove P(0) **Base Case**

2. Let k be an arbitrary integer ≥ 0
   3. Assume that P(k) is true **Inductive Hypothesis**
   4. ...
   5. Prove P(k+1) is true **Inductive Step**

6. P(k) → P(k+1) Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) Intro ∀ from 2-6
8. ∀ n P(n) Induction Rule 1&7

Conclusion

Proof:

1. "We will show that P(n) is true for every $n \geq 0$ by **induction**."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

   Assume P(k) is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Want to prove that P(k+1) is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !)
5. "Conclusion: Result follows by induction."

prove: $n^n \geq n!$ for all $n \geq 1$

prove $3^n \geq n^2$ for all $n \geq 3$.

prove $3^n \geq n^2$ for all $n \geq 3$.

Let $P(n)$ be "$3^n \geq n^2$" for all $n \geq 3$.

We go by induction on n.

Base Case:

$3^3 = 27 \geq 9 = 3^2$. So, $P(3)$ is true.

Induction Hypothesis:

Suppose $P(k)$ is true for some arbitrary $k \geq 3$.

Induction Step:

Note that $3^{k+1} = 3(3^k) \geq 3(k^2)$, by the IH.

Furthermore, note that $(k+1)^2 = k^2 + 2k + 1$.

Note that since $k \geq 3$, $k^2 \geq 3k \geq 2k$. And similarly, $k^2 \geq 1$.

So, continuing from above:

$$3^{k+1} = 3(3^k) \geq 3(k^2) = k^2 + k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$$

Since this is exactly $P(k+1)$, we've shown $P(k) \rightarrow P(k+1)$

Thus, $P(n)$ is true for all $n \geq 3$, by induction.

prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Note that $2(n+1)^3 = 2n^3 + 6n^2 + 6n + 2$.

Let P(n) be "$2n^3 + 2n - 5 \geq n^2$" for all $n \geq 2$.

We go by induction on n.

Base Case:

$2*2^3 + 2*2 - 5 = 45 \geq 4 = 2^2$. So, P(0) is true.

Induction Hypothesis:

Suppose P(n) is true for some arbitrary $n \geq 2$.

Induction Step: Then, note that…

$(n+1)^2$	$\leq$	$n^2 + 2n + 1$	
	$\leq$	$(2n^3 + 2n - 5) + 2n + 1$	(by IH)
	$\leq$	$(2n^3 + 4n + 1) - 5$	(Re-arranging)
	$\leq$	$(2n^3 + 6n^2 + 6n + 2) - 5$	$(4n + 1 \leq 6n + 6n^2 + 2)$
	$\leq$	$2(n+1)^3 - 5$	(Factoring)
	$\leq$	$2(n+1)^3 + 2n - 5$	$(0 \leq 2n)$

Since this is exactly P(k+1), we've shown P(k) → P(k+1)

Thus, P(n) is true for all $n \geq 3$, by induction.

strong induction

$$P(0)$$

$$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \rightarrow P(k+1)\Big)$$

$$\therefore\ \forall n\ P(n)$$

Follows from ordinary induction applied to
$Q(n) = P(0) \land P(1) \land P(2) \land \cdots \land P(n)$

strong induction English proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

every integer at least 2 is the product of primes

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We argue by strong induction.

P(n) = "n can be expressed as a product of primes" for $n \geq 2$.

Base Case:

Note that 2 is prime; so, we can express it as "2" which is a product of primes.

Induction Hypothesis:

Suppose $P(2) \wedge P(3) \wedge \cdot \cdot \cdot \wedge P(k)$ is true for some $k \geq 2$.

Induction Step:

We go by cases.

Suppose k+1 is prime. Then, "k+1" is a product of primes.

Suppose k+1 is composite. Then, $k+1 = ab$ for some a and b such that $1 < a, b < k+1$.

By our IH, we know $a = p_1 p_2 \cdots p_m$ and $b = q_1 q_2 \cdots q_n$.

So, $k+1 = ab =$ "$p_1 p_2 \cdots p_m q_1 q_2 \cdots q_n$", which is a product of primes.

Thus, our claim is true for $n \geq 2$ by strong induction.

recursive definition of functions

- $F(0) = 0;\ F(n+1) = F(n) + 1$ **for all** $n \geq 0$
- $G(0) = 1;\ G(n+1) = 2 \times G(n)$ **for all** $n \geq 0$
- $0! = 1;\ (n+1)! = (n+1) \times n!$ **for all** $n \geq 0$
- $H(0) = 1;\ H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

bounding the Fibonacci numbers

Theorem: $f_n < 2^n$ **for all** $n \geq 2$.

bounding the Fibonacci numbers

Theorem: $2^{\frac{n}{2}-1} \leq f_n$ for all $n \geq 2$