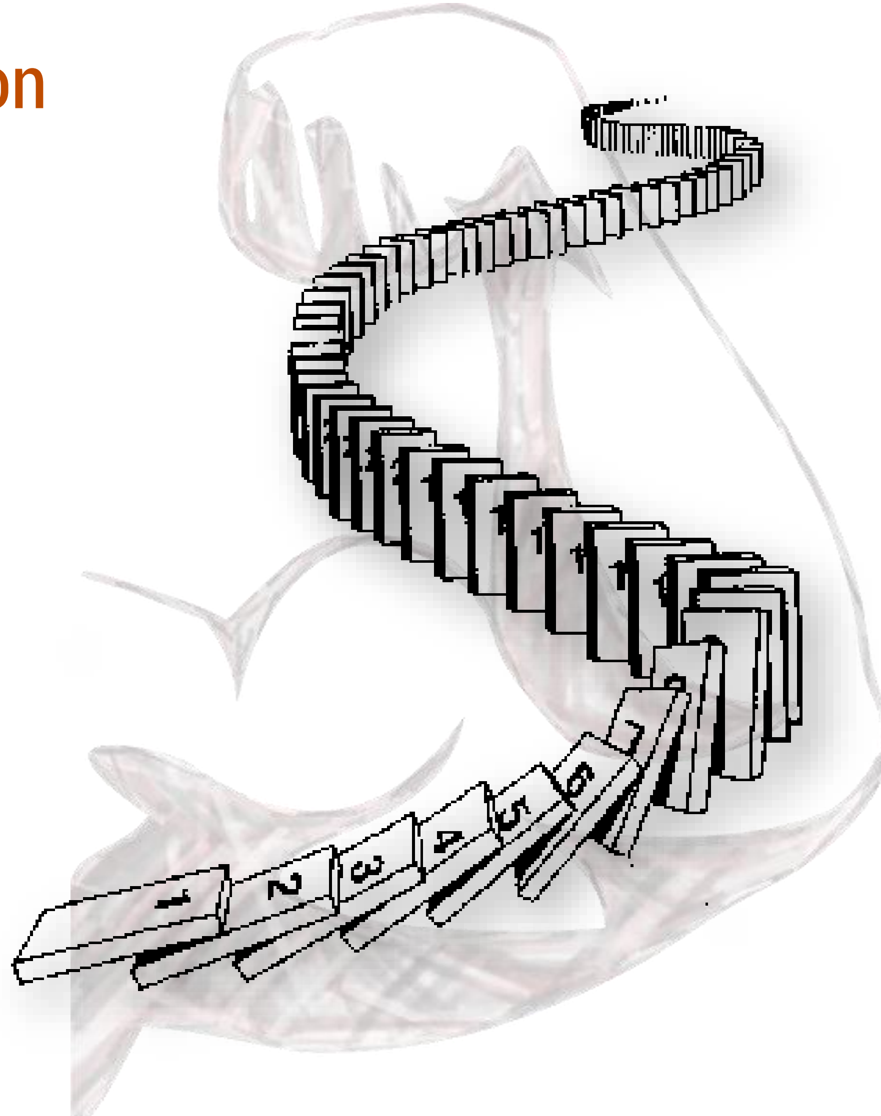
Spring 2015

Lecture 16: **Strong** induction



review: **induction** is a rule of inference

Domain: Natural Numbers

$$P(0)$$
$$\forall\, k\, (P(k) \rightarrow P(k+1))$$

$$\therefore \forall\, n\, P(n)$$

review: using the induction rule in a formal proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \rightarrow P(k+1)) \end{array}}{\therefore \forall\, n\, P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
   3. Assume that P(k) is true
   4. ...
   5. Prove P(k+1) is true
6. $P(k) \rightarrow P(k+1)$ Direct Proof Rule
7. $\forall$ k $(P(k) \rightarrow P(k+1))$ Intro $\forall$ from 2-6
8. $\forall$ n P(n) Induction Rule 1&7

review: format of an induction proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \;\rightarrow\; P(k+1)) \end{array}}{\therefore\, \forall\, n\, P(n)}$$

1. Prove P(0) — **Base Case**
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true — **Inductive Hypothesis**
    4. ...
    5. Prove P(k+1) is true — **Inductive Step**
6. $P(k) \rightarrow P(k+1)$ — Direct Proof Rule
7. $\forall\, k\, (P(k) \rightarrow P(k+1))$ — Intro $\forall$ from 2-6
8. $\forall\, n\, P(n)$ — Induction Rule 1&7

Conclusion

Proof:

1. "We will show that P(n) is true for every n ≥ 0 by **induction**."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !)
5. "Conclusion: Result follows by induction."

Therefore

horses

Thm: All horses are the same color.

P(n) = "Every collection of n horses has the same color."

Base case: P(1) true.

IH: Assume P(k) for an arbitrary natural # k

IS: Consider k+1 horses

same color

1 2 3 .... k k+1

same color

⟹ All k+1 horses ... ⟹ P(k+1)

k = 1

$\boxed{1}$ $\boxed{2}$

prove $3^n \geq n^2$ for all $n \geq 3$. 1

$P(n) =$ "$3^n \geq n^2$"

Base case: $P(1)$ $3^1 \geq 1^2$ true.

$P(2)$ $3^2 \geq 2^2$ true.

IH: Assume $3^k \geq k^2$ for some $k \geq 2$

IS: $3^k \geq k^2$

$\Rightarrow 3 \cdot 3^k \geq 3k^2$

$3^{k+1} =$

$= k^2 + k^2 + k^2$

$\geq k^2 + 2k + 1$ (since $k \geq 2$)

$= (k+1)^2$

prove $3^n \geq n^2$ for all $n \geq 3$.

Let P(n) be "$3^n \geq n^2$" for all $n \geq 3$.

We go by induction on n.

Base Case:

$3^3 = 27 \geq 9 = 3^2$. So, P(3) is true.

Induction Hypothesis:

Suppose P(k) is true for some arbitrary $k \geq 3$.

Induction Step:

Note that $3^{k+1} = 3(3^k) \geq 3(k^2)$, by the IH.

Furthermore, note that $(k+1)^2 = k^2 + 2k + 1$.

Note that since $k \geq 3$, $k^2 \geq 3k \geq 2k$. And similarly, $k^2 \geq 1$.

So, continuing from above:

$$3^{k+1} = 3(3^k) \geq 3(k^2) = k^2 + k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$$

Since this is exactly P(k+1), we've shown $P(k) \to P(k+1)$

Thus, P(n) is true for all $n \geq 3$, by induction.

prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Note that $2(n+1)^3 = 2n^3 + 6n^2 + 6n + 2$.

Let P(n) be "$2n^3 + 2n - 5 \geq n^2$" for all $n \geq 2$.

We go by induction on n.

Base Case:

$2*2^3 + 2*2 - 5 = 45 \geq 4 = 2^2$. So, ~~P(0)~~ P(2) is true.

Induction Hypothesis:

Suppose P(n) is true for some arbitrary $n \geq 2$.

Induction Step: Then, note that…

$$
\begin{aligned}
(n+1)^2 &\leq n^2 + 2n + 1 & & \\
&\leq (2n^3 + 2n - 5) + 2n + 1 & & \text{(by IH)} \\
&\leq (2n^3 + 4n + 1) - 5 & & \text{(Re-arranging)} \\
&\leq (2n^3 + 6n^2 + 6n + 2) - 5 & & (4n + 1 \leq 6n + 6n^2 + 2) \\
&\leq 2(n+1)^3 - 5 & & \text{(Factoring)} \\
&\leq 2(n+1)^3 + 2n - 5 & & (0 \leq 2n)
\end{aligned}
$$

Since this is exactly P(k+1), we've shown P(k) → P(k+1)

Thus, P(n) is true for all ~~$n \geq 3$~~ $n \geq 2$, by induction.

$$P(0)$$
$$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \to P(k+1)\Big)$$

$$\therefore\ \forall n\ P(n)$$

Follows from ordinary induction applied to

$$Q(n) = P(0) \land P(1) \land P(2) \land \cdots \land P(n)$$

strong induction English proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

every integer at least 2 is the product of primes

$P(n) =$ "n is a product of primes"

Base case: $P(2)$ is true b/c $2 = 2$
and 2 is prime

IH: For some $k \geq 2$, $P(j)$ holds for
all $2 \leq j \leq k$. $(P(2) \wedge P(3) \wedge \dots \wedge P(k))$

IS: If $k+1$ is prime, then $P(k+1)$ holds

If $k+1$ is not prime $k+1 = ab$
with $1 < a < k+1 \Rightarrow P(a)$
$1 < b < k+1 \Rightarrow P(b)$ by IH

every integer at least 2 is the product of primes

$k+1 = ab$

$P(a) \Rightarrow a = p_1 p_2 \cdots p_r$ $\{p_i\}$ primes

$P(b) \Rightarrow b = q_1 q_2 \cdots q_s$ $\{q_i\}$ primes

$\Rightarrow k+1 = p_1 \cdots p_r q_1 q_2 \cdots q_r$

$\Rightarrow P(k+1).$

every integer at least 2 is the product of primes

We argue by strong induction.
$P(n)$ = "n can be expressed as a product of primes" for $n \geq 2$.

Base Case:

Note that 2 is prime; so, we can express it as "2" which is a product of primes.

Induction Hypothesis:

Suppose $P(2) \wedge P(3) \wedge \cdot \cdot \cdot \wedge P(k)$ is true for some $k \geq 2$.

Induction Step:

We go by cases.
Suppose $k+1$ is prime. Then, "$k+1$" is a product of primes.
Suppose $k+1$ is composite. Then, $k+1 = ab$ for some a and b such that $1 < a, b < k+1$.
By our IH, we know $a = p_1p_2 \cdots p_m$ and $b = q_1q_2 \cdots q_n$.
So, $k+1 = ab =$ "$p_1p_2 \cdots p_mq_1q_2 \cdots q_n$", which is a product of primes.

Thus, our claim is true for $n \geq 2$ by strong induction.

recursive definition of functions

- $F(0) = 0;\ F(n+1) = F(n) + 1$ **for all** $n \geq 0$

  $F(n) = n$

- $G(0) = 1;\ G(n+1) = 2 \times G(n)$ **for all** $n \geq 0$

  $G(n) = 2^n$

- $0! = 1;\ (n+1)! = (n+1) \times n!$ **for all** $n \geq 0$

- $H(0) = 1;\ H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$

  $H(n) = 2^{2^{2^{2^{2}}}}$ n times

Fibonacci numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$ for all $n \geq 2$

$f_2 = f_1 + f_0 = 1 + 0 = 1$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

...

$f_0 = 0$ $\quad f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$
$f_1 = 1$

bounding the Fibonacci numbers

Theorem: $f_n < 2^n$ **for all** $n \geq 2$.

Proof by induction.

Base case: $f_0 = 0 < 2^0 = 1$ ✓

$f_1 = 1 < 2^1 = 2$ ✓

IH: Assume $f_j < 2^j$ for all $0 \leq j \leq k$

for some $k \geq 1$.

Since $k+1 \geq 2$, $f_{k+1} = f_k + f_{k-1}$

$\overset{(IH)}{<} 2^k + 2^{k-1}$

...

FRI

Theorem: $2^{\frac{n}{2}-1} \leq f_n$ for all $n \geq 2$