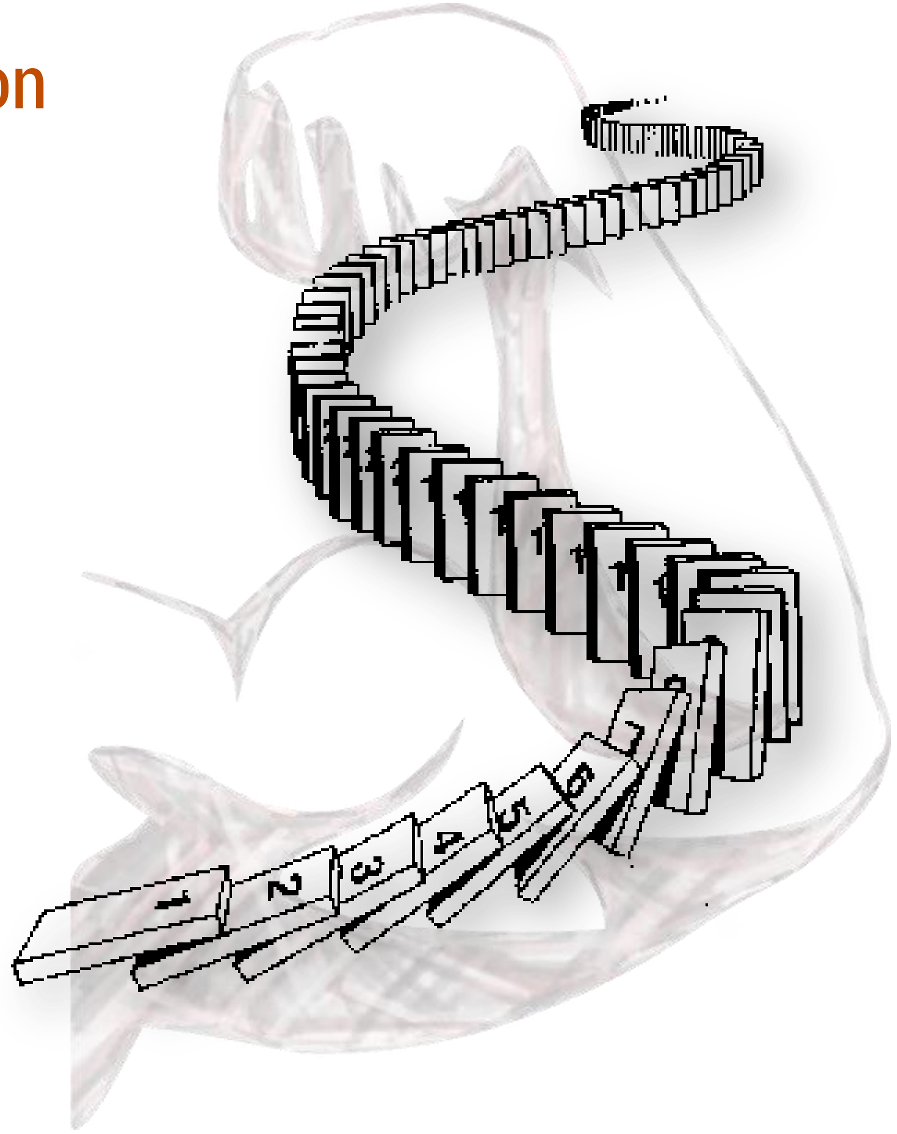
Spring 2015

Lecture 16: **Strong** induction



review: **induction** is a rule of inference

Domain: Natural Numbers

$$P(0)$$
$$\forall\, k\, (P(k) \rightarrow P(k+1))$$

$$\therefore \forall\, n\, P(n)$$

review: using the induction rule in a formal proof

$$\frac{P(0) \qquad \forall k\,(P(k) \rightarrow P(k+1))}{\therefore \forall n\, P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. $P(k) \rightarrow P(k+1)$ — Direct Proof Rule
7. $\forall k\,(P(k) \rightarrow P(k+1))$ — Intro $\forall$ from 2-6
8. $\forall n\, P(n)$ — Induction Rule 1&7

review: format of an induction proof

$$P(0)$$
$$\forall\, k\, (P(k) \rightarrow P(k+1))$$
$$\therefore \forall\, n\, P(n)$$

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true — **Inductive Hypothesis**
    4. ...
    5. Prove P(k+1) is true — **Inductive Step**

6. P(k) → P(k+1)   Direct Proof Rule
7. ∀ k (P(k) → P(k+1))   Intro ∀ from 2-6
8. ∀ n P(n)   Induction Rule 1&7

Conclusion

review: inductive proof in five easy steps

Proof:

1. "We will show that P(n) is true for every n ≥ 0 by **induction**."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !)
5. "Conclusion: Result follows by induction."

prove: $n^n \geq n!$ for all $n \geq 1$

$P(n) =$ "$n^n \geq n!$" $n! = n(n-1)\cdots 2\cdot 1$

Base Case: $1^1 \geq 1!$ $P(1)$ holds

IH: $P(k)$ holds for some $k \geq 1$.

IS: Want to prove $P(k+1)$

$$(k+1)^{k+1} = (k+1)(k+1)^k \geq (k+1)k^k \underset{\uparrow \text{ IH}}{\geq} (k+1)k! = (k+1)!$$

$P(k+1)$ is true

Conclusion $P(n)$ holds for $n \geq 1$.

IS: $(k+1)^{k+1} \overset{?}{\geq} (k+1)!$

$\Updownarrow$

$(k+1)^k \geq k!$

$k^k \geq k!$

...

prove $3^n \geq n^2$ for all $n \geq 3$.

$P(n) = $ " $3^n \geq n^2$ " Goal: $P(n)$ holds for $n \geq 3$.

Base Case: $3^3 \geq 3^2$ $P(3)$ holds

I.H. $P(k)$ holds for some $k \geq 3$

I.S.: Show $P(k+1)$ holds.

$$3^{k+1} \overset{?}{\geq} (k+1)^2$$

$$3^{k+1} = 3 \cdot 3^k \geq 3 \cdot k^2$$

$$(k+1)^2 = k^2 + 2k + 1$$

$2k \leq k^2$ since $k \geq 3$ $\Rightarrow 2k+1 \leq 2k^2$

$1 \leq k^2$ since $k \geq 1$

$$(k+1)^2 \leq 3k^2 \leq 3^{k+1}$$

So $P(k+1)$ holds

prove $3^n \geq n^2$ for all $n \geq 3$.

Let $P(n)$ be "$3^n \geq n^2$" for all $n \geq 3$.

We go by induction on n.

Base Case:

$3^3 = 27 \geq 9 = 3^2$. So, $P(3)$ is true.

Induction Hypothesis:

Suppose $P(k)$ is true for some arbitrary $k \geq 3$.

Induction Step:

Note that $3^{k+1} = 3(3^k) \geq 3(k^2)$, by the IH.

Furthermore, note that $(k+1)^2 = k^2 + 2k + 1$.

Note that since $k \geq 3$, $k^2 \geq 3k \geq 2k$. And similarly, $k^2 \geq 1$.

So, continuing from above:

$$3^{k+1} = 3(3^k) \geq 3(k^2) = k^2 + k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$$

Since this is exactly $P(k+1)$, we've shown $P(k) \rightarrow P(k+1)$

Thus, $P(n)$ is true for all $n \geq 3$, by induction.

prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Note that $2(n+1)^3 = 2n^3 + 6n^2 + 6n + 2$.

Let P(n) be "$2n^3 + 2n - 5 \geq n^2$" for all $n \geq 2$.

We go by induction on n.

Base Case:

$2*2^3 + 2*2 - 5 = 45 \geq 4 = 2^2$. So, P(0) is true.

Induction Hypothesis:

Suppose P(n) is true for some arbitrary $n \geq 2$.

Induction Step: Then, note that…

$(n+1)^2$	$\leq$	$n^2 + 2n + 1$	
	$\leq$	$(2n^3 + 2n - 5) + 2n + 1$	(by IH)
	$\leq$	$(2n^3 + 4n + 1) - 5$	(Re-arranging)
	$\leq$	$(2n^3 + 6n^2 + 6n + 2) - 5$	$(4n + 1 \leq 6n + 6n^2 + 2)$
	$\leq$	$2(n+1)^3 - 5$	(Factoring)
	$\leq$	$2(n+1)^3 + 2n - 5$	$(0 \leq 2n)$

Since this is exactly P(k+1), we've shown P(k) → P(k+1)

Thus, P(n) is true for all $n \geq 3$, by induction.

strong induction

$$P(0)$$

$$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \to P(k+1)\Big)$$

$$\therefore \forall n\ P(n)$$

Follows from ordinary induction applied to

$$Q(n) = P(0) \land P(1) \land P(2) \land \cdots \land P(n)$$

strong induction English proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

every integer at least 2 is the product of primes

$P(n)$ = "n can be written as product of primes".

Base Case: 2 can be written as "2".

IH: For some $k \geq 2$
$P(2) \wedge \cdots \wedge P(k)$ holds.

IS: Show $P(k+1)$ holds.

Case 1: $k+1$ is a prime. We write is as "$k+1$".

Case 2: It is a composite. $k+1 = a \cdot b$ where $2 \leq a, b \leq k$

$P(a)$ holds $a = p_1 \cdots p_i$
$P(b)$ holds $b = q_1 \cdots q_j$

$k+1 = ab = p_1 \cdots p_i \cdot q_1 \cdots q_j$

$P(k+1)$ is true

$p(n)$ holds for $n \geq 2$.

every integer at least 2 is the product of primes

We argue by strong induction.
P(n) = "n can be expressed as a product of primes" for $n \geq 2$.

Base Case:

Note that 2 is prime; so, we can express it as "2" which is a product of primes.

Induction Hypothesis:

Suppose $P(2) \wedge P(3) \wedge \cdot \cdot \cdot \wedge P(k)$ is true for some $k \geq 2$.

Induction Step:

We go by cases.
Suppose k+1 is prime. Then, "k+1" is a product of primes.
Suppose k+1 is composite. Then, k+1 = ab for some a and b such that $1 < a, b < k+1$.
By our IH, we know $a = p_1 p_2 \cdots p_m$ and $b = q_1 q_2 \cdots q_n$.
So, k+1 = ab = "$p_1 p_2 \cdots p_m q_1 q_2 \cdots q_n$", which is a product of primes.

Thus, our claim is true for $n \geq 2$ by strong induction.

recursive definition of functions

- $F(0) = 0;\ F(n+1) = F(n) + 1$ **for all** $n \geq 0$

  $F(n) = n$

- $G(0) = 1;\ G(n+1) = 2 \times G(n)$ **for all** $n \geq 0$

  $G(n) = 2^n$

- $0! = 1;\ (n+1)! = (n+1) \times n!$ **for all** $n \geq 0$

- $H(0) = 1;\ H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$

  $H(n) = \underbrace{2^{2^{2^{\cdots}}}}_{n}$

  $H(1) = 2$

  $H(2) = 4$

  $H(3) = 16$

  $H(4) = 65\ldots$

  $H(5) = 2^{65\ldots}$

Fibonacci numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$ **for all** $n \geq 2$

$f_2 = 1$

$f_3 = 2$

$f_4 = 3$

f

bounding the Fibonacci numbers

Theorem: $f_n < 2^n$ **for all** $n \geq 2$.

$P(n)$: "$f_n < 2^n$".

Base Case: $P(2)$ $\quad f_2 = 1 < 2^2 = 4$

IH: For some $k \geq 2$, $P(j)$ holds for all $2 \leq j \leq k$.

IS: $f_{k+1} = f_k + f_{k-1} < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$

$f_k < 2^k$

$f_{k-1} < 2^{k-1}$

$P(k+1)$ holds

$P(3)$: $\quad f_3 = 2 < 2^3$

bounding the Fibonacci numbers

Theorem: $2^{\frac{n}{2}-1} \leq f_n$ for all $n \geq 2$