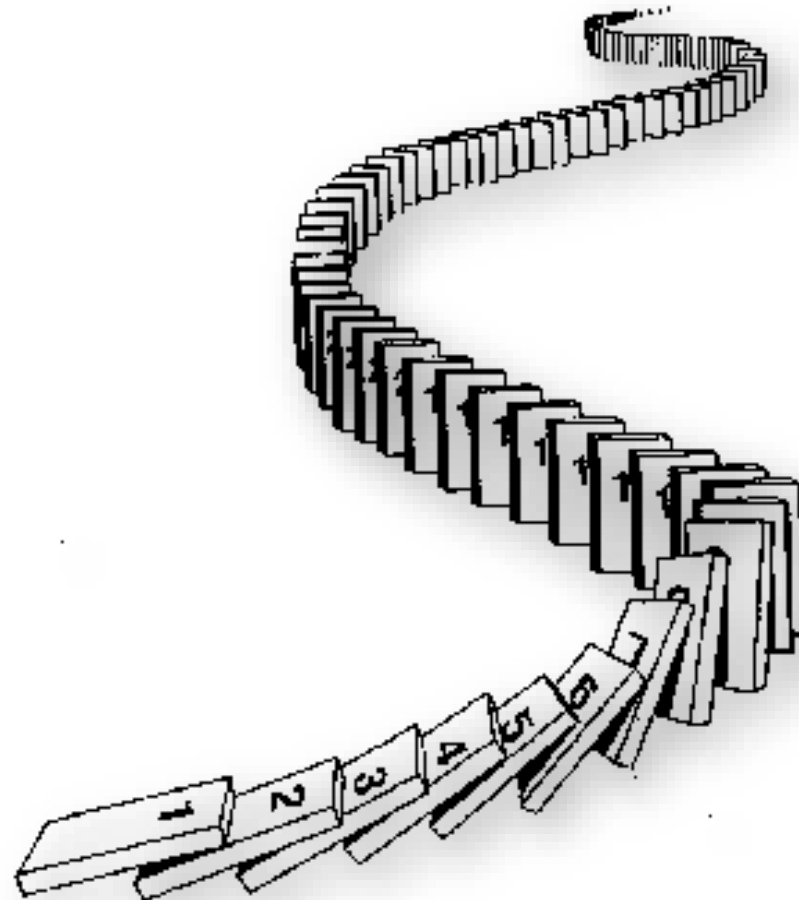


cse 311: foundations of computing

Fall 2015

Lecture 15: Induction



prove: for all $n > 0$, a is odd $\rightarrow a^n$ is odd

Let $n > 0$ be arbitrary.

Suppose that a is odd. We know that if a, b are odd, then ab is also odd.

So: $(\cdots ((a \cdot a) \cdot a) \cdots \cdot a) = a^n$ [n times]

Those “...”s are a problem! We’re trying to say “we can use the same argument over and over...”

We’ll come back to this.

Method for proving statements about all integers ≥ 0

- A new logical inference rule!
 - It only applies over the natural numbers
 - The idea is to **use** the special structure of the naturals to prove things more easily
- Particularly useful for reasoning about programs!

```
for(int i=0; i < n; n++) { ... }
```

- Show $P(i)$ holds after i times through the loop

```
public int f(int x) {  
    if (x == 0) { return 0; }  
    else { return f(x-1)+1; }  
}
```

- $f(x) = x$ for all values of $x \geq 0$ naturally shown by induction.

induction is a rule of inference

Domain: Natural Numbers

$$\begin{array}{l} P(0) \\ \forall k (P(k) \rightarrow P(k + 1)) \end{array}$$

$$\therefore \forall n P(n)$$

using the induction rule in a formal proof

$$\begin{array}{l} P(0) \\ \forall k (P(k) \rightarrow P(k + 1)) \end{array}$$

$$\therefore \forall n P(n)$$

1. Prove $P(0)$
2. Let k be an arbitrary integer ≥ 0
 3. Assume that $P(k)$ is true
 4. ...
 5. Prove $P(k+1)$ is true
6. $P(k) \rightarrow P(k+1)$
7. $\forall k (P(k) \rightarrow P(k+1))$
8. $\forall n P(n)$

Direct Proof Rule

Intro \forall from 2-6

Induction Rule 1&7

format of an induction proof

$$P(0)$$

$$\forall k (P(k) \rightarrow P(k + 1))$$

$$\therefore \forall n P(n)$$

1. Prove $P(0)$

Base Case

2. Let k be an arbitrary integer ≥ 0

3. Assume that $P(k)$ is true

Inductive Hypothesis

4. ...

5. Prove $P(k+1)$ is true

Inductive Step

6. $P(k) \rightarrow P(k+1)$

Direct Proof Rule

7. $\forall k (P(k) \rightarrow P(k+1))$

Intro \forall from 2-6

8. $\forall n P(n)$

Induction Rule 1&7

Conclusion

$$1 + 2 + 4 + 8 + \dots + 2^n$$

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

Can we describe the pattern?

$$1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$$

proving $1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$

- We could try proving it normally...
 - We want to show that $1 + 2 + 4 + \dots + 2^n = 2^{n+1}$.
 - So, what do we do now? We can sort of explain the pattern, but that's not a proof...
- We could prove it for $n=1, n=2, n=3, \dots$
(individually), but that would literally take forever...

inductive proof in five easy steps

Proof:

1. “We will show that $P(n)$ is true for every $n \geq 0$ by **induction**.”

2. “Base Case:” Prove $P(0)$

3. “Inductive Hypothesis:”

Assume $P(k)$ is true for some arbitrary integer $k \geq 0$ ”

4. “Inductive Step:” Want to prove that $P(k+1)$ is true:

Use the goal to figure out what you need.

Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!)

5. “Conclusion: Result follows by induction.”

proving $1 + 2 + \dots + 2^n = 2^{n+1} - 1$

proving $1 + 2 + \dots + 2^n = 2^{n+1} - 1$

1. Let $P(n)$ be " $1 + 2 + \dots + 2^n = 2^{n+1} - 1$ ". We will show $P(n)$ is true for all natural numbers by induction.
2. Base Case ($n=0$): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$
3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary $k \geq 0$.
4. Induction Step:

Goal: Show $P(k+1)$, i.e. show $1 + 2 + \dots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$$1 + 2 + \dots + 2^k = 2^{k+1} - 1 \quad \text{by IH}$$

Adding 2^{k+1} to both sides, we get:

$$1 + 2 + \dots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

So, we have $1 + 2 + \dots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly $P(k+1)$.

5. Thus $P(k)$ is true for all $k \in \mathbb{N}$, by induction.

another example of a pattern

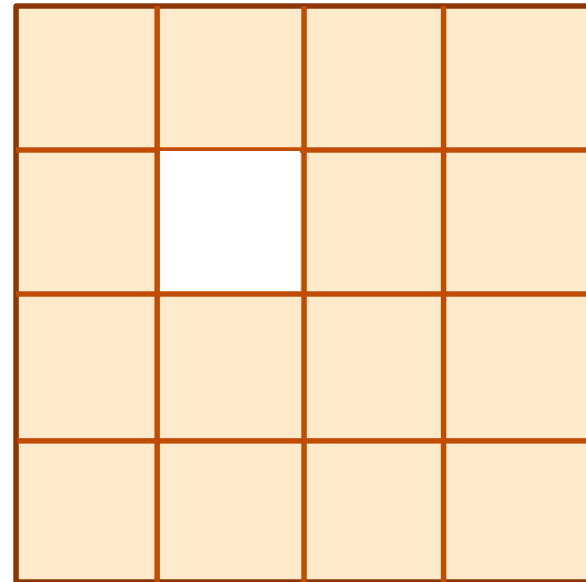
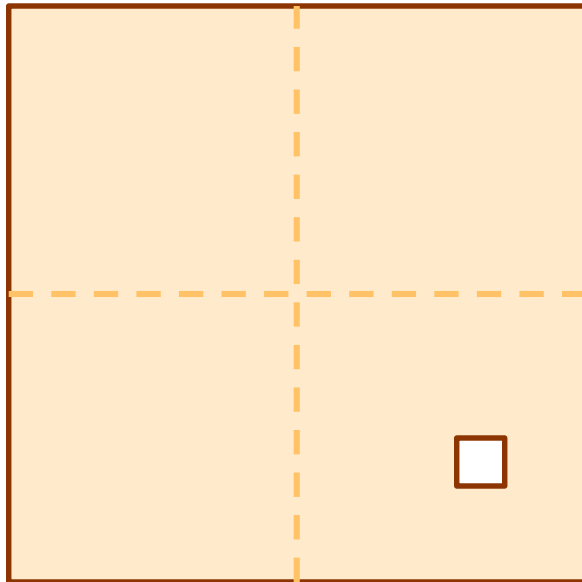
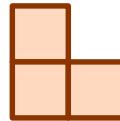
- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- ...

prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

$$\text{For all } n \geq 1: 1 + 2 + \cdots + n = \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

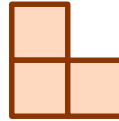
checkerboard tiling

Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:



checkerboard tiling

Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:



prove: $n^n \geq n!$ for all $n \geq 1$
