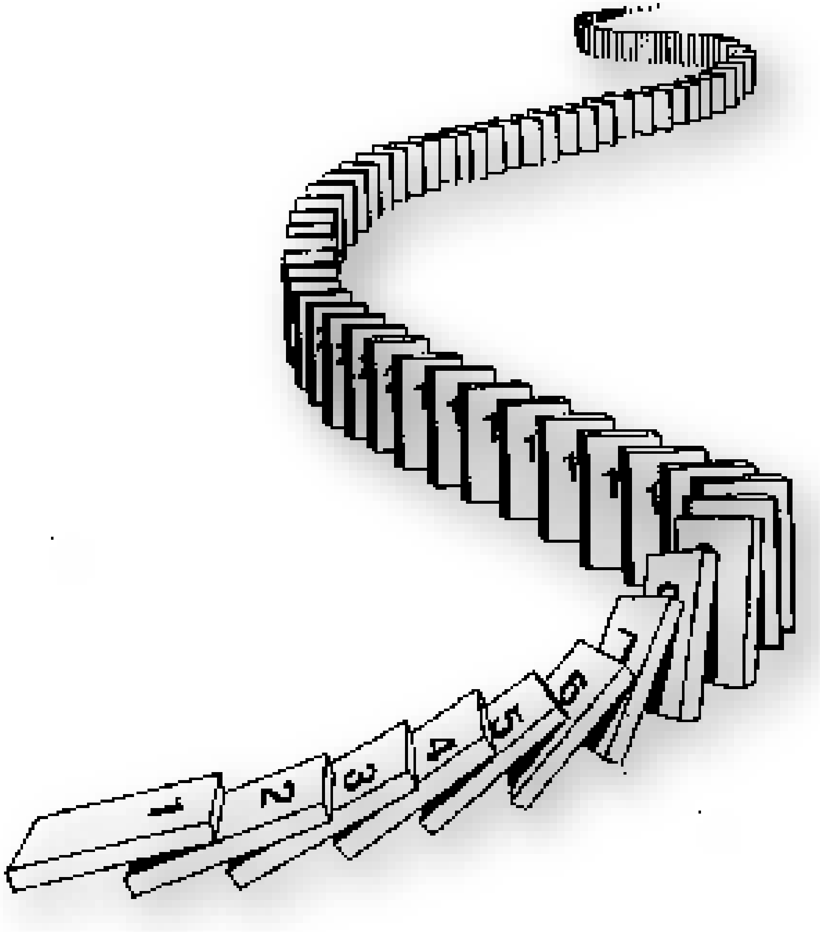
Fall 2015

Lecture 15: Induction



prove: for all $n > 0$, a is odd $\rightarrow a^n$ is odd

Let $n > 0$ be arbitrary.

Suppose that a is odd. We know that if a, b are odd, then ab is also odd.

So: $\left(\cdots\left((a \cdot a) \cdot a\right) \cdots \cdot a\right) = a^n$ [n times]

Those "$\cdots$"s are a problem! We're trying to say "we can use the same argument over and over…"

We'll come back to this.

Method for proving statements about all integers ≥ 0

- A new logical inference rule!
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- Particularly useful for reasoning about programs!

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
        else { return f(x-1)+1; }}
```

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

induction is a rule of inference

Domain: Natural Numbers

Q

Q(k) holds for all even k

$$P(0)$$
$$\forall\, k\, (P(k) \;\rightarrow\; P(k+1))$$

$$\therefore \forall\, n\, P(n)$$

$P(k) \overset{def}{=} Q(2k)$

$\dfrac{P(0),\ \forall k\, P(k) \rightarrow P(k+1)}{\forall n\ P(n)} \Rightarrow \forall n\, Q(2n)$

using the induction rule in a formal proof

$$\frac{P(0) \qquad \forall k\, (P(k) \rightarrow P(k+1))}{\therefore \forall n\, P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. $P(k) \rightarrow P(k+1)$ — Direct Proof Rule
7. $\forall k\, (P(k) \rightarrow P(k+1))$ — Intro $\forall$ from 2-6
8. $\forall n\, P(n)$ — Induction Rule 1&7

format of an induction proof

$$P(0)$$
$$\forall\, k\ (P(k)\ \rightarrow\ P(k+1))$$
$$\therefore\ \forall\, n\ P(n)$$

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0
   3. Assume that P(k) is true

Inductive Hypothesis

4. ...
5. Prove P(k+1) is true

Inductive Step

6. $P(k) \rightarrow P(k+1)$	Direct Proof Rule
7. $\forall$ k $(P(k) \rightarrow P(k+1))$	Intro $\forall$ from 2-6
8. $\forall$ n P(n)	Induction Rule 1&7

Conclusion

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

$1 + 2 + \cdots + 2^n = 2^{n+1} - 1$

Can we describe the pattern?

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$$

proving $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$

- We could try proving it normally…
  - We want to show that $1 + 2 + 4 + \cdots + 2^n = 2^{n+1}$.
  - So, what do we do now? We can sort of explain the pattern, but that's not a proof…
- We could prove it for n=1, n=2, n=3, … (individually), but that would literally take forever…

inductive proof in five easy steps

Proof:

1. "We will show that P(n) is true for every $n \geq 0$ by **induction**."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

   Assume P(k) is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Want to prove that P(k+1) is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it.
   (Don't assume P(k+1) !)
5. "Conclusion: Result follows by induction."

proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

$P(n) =$ " $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ "

Goal: $\forall n\ P(n)$ Domain: nat. numbers

Base case: $P(0)$ is " $1 = 2^1 - 1 = 1$ " true.

Inductive hypoth: Assume $P(k)$ for some $k \geq 0$

Ind. step: By $P(k)$, we know

$$1 + 2 + \cdots + 2^k = 2^{k+1} - 1$$

$$\Rightarrow 1 + 2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

$\Rightarrow P(k+1)$.

By induction $\forall k P(k)$.

proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

1. Let P(n) be "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". We will show P(n) is true for all natural numbers by induction.
2. Base Case (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$
3. Induction Hypothesis: Suppose that P(k) is true for some arbitrary $k \geq 0$.
4. Induction Step:

   Goal: Show P(k+1), i.e. show $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

   $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ by IH

   Adding 2^{k+1} to both sides, we get:

   $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly P(k+1).
5. Thus P(k) is true for all $k \in \mathbb{N}$, by induction.

another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- …

$3 \mid 2^{2k} - 1$

By induction $\forall n\, P(n)$ — prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

$P(n) =$ " $3 \mid 2^{2n} - 1$ "

Goal: $\forall n\, P(n)$ domain: $n \geq 0$ integer

Base case: $P(0) \equiv$ " $3 \mid 2^0 - 1$ "

$\equiv$ " $3 \mid 0$ " true b/c $0 = 0 \cdot 3$

IH: Assume that $3 \mid 2^{2k} - 1$ for some $k \geq 0$

IS: $2^{2k} - 1 = 3a$ for some int a by IH.

$\Rightarrow 4(2^{2k} - 1) = 12a$ $3 \mid 2^{2(k+1)} - 1$

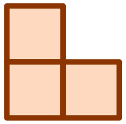
$\Uparrow$

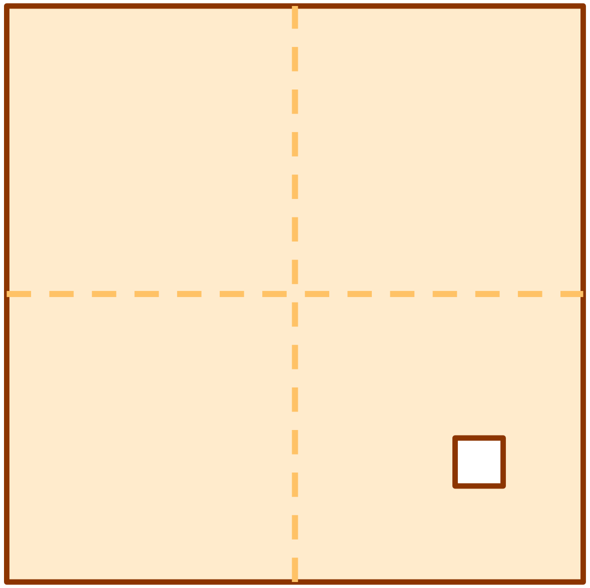
$\Rightarrow 2^2(2^{2k} - 1) = 2^{2(k+1)} - 4 = 12a$

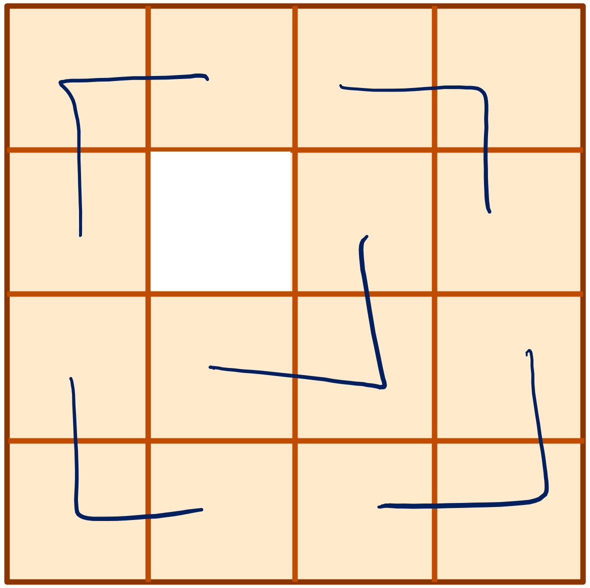
$\Rightarrow 2^{2(k+1)} - 1 = 12a + 3 = 3(4a + 1)$

For all $n \geq 1$: $1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

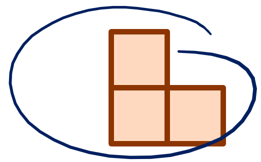
Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:





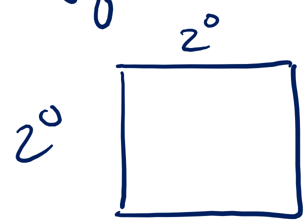


Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:



$P(n)$ = tiles $2^n \times 2^n$ ch. b. one square missing

Base case: $P(0)$



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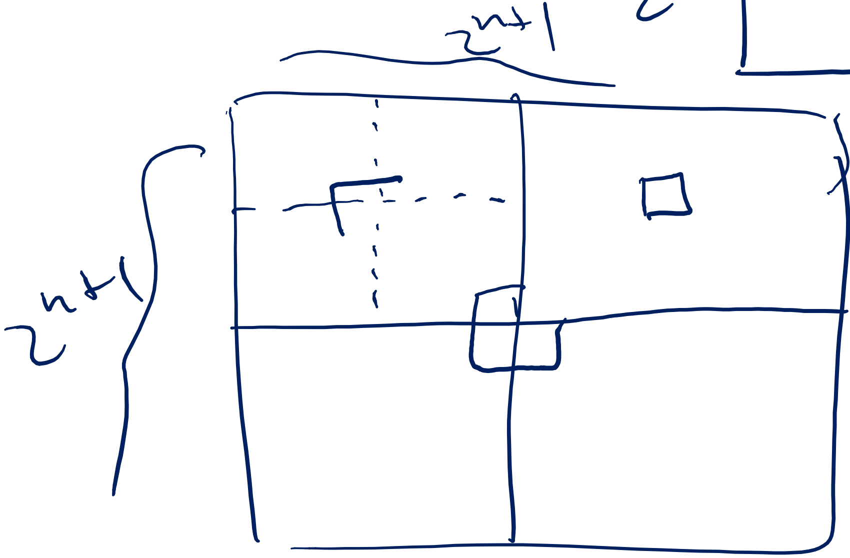
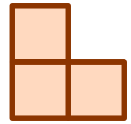


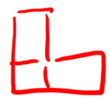
$2^n \times 2^n$

$\Downarrow$

$2^{n+1} \times 2^{n+1}$

Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:



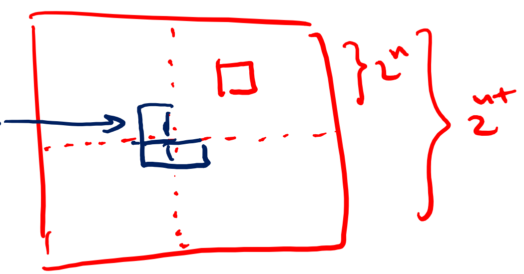
P(n) = "A $2^n \times 2^n$ checkerboard w/ one square removed can be tiled by ."

Base case: P(0) We just have an empty 1 x 1 board which is tiled w/out doing anything.

IH: Assume P(n) holds for some integer $n \geq 0$.

IS: Consider a $2^{n+1} \times 2^{n+1}$ board with one square removed.

By symmetry, we can assume:



Now place a block at the intersection

We now have four $2^n \times 2^n$ squares S_1, S_2, S_3, S_4 each with one block removed. By IH, each of S_1, S_2, S_3, S_4 can be tiled.

(cont.) Thus the $2^{n+1} \times 2^{n+1}$ square can be tiled.

$\Rightarrow P(n+1)$

prove: $n^n \geq n!$ for all $n \geq 1$