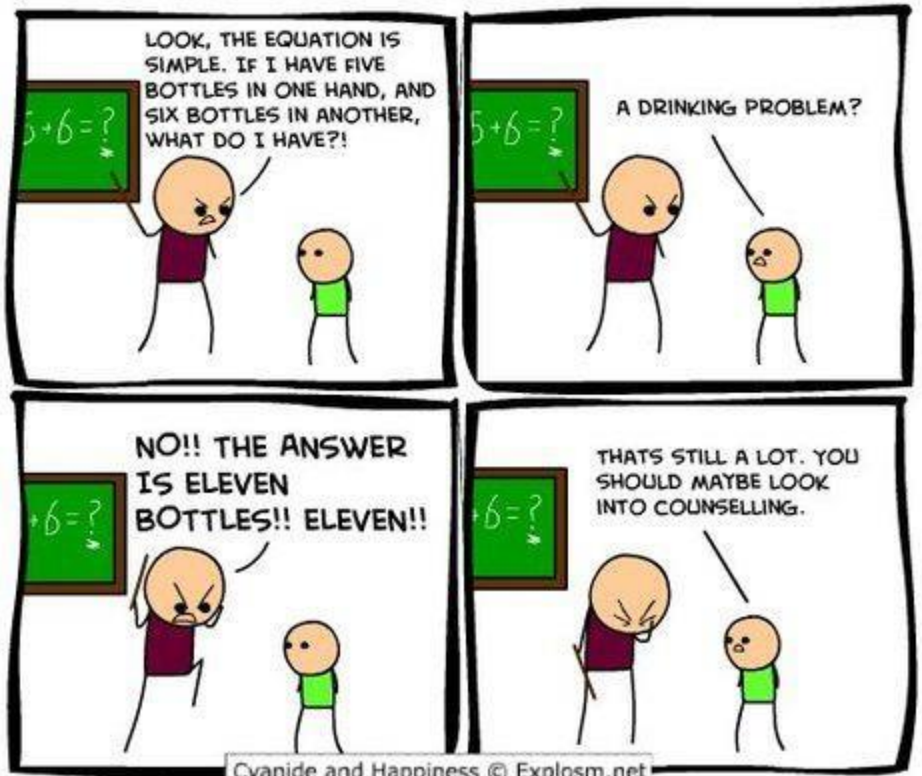
Fall 2015
Lecture 14: Modular congruences



Useful GCD Fact

If a and b are positive integers, then

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

Proof:

By definition $a = (a \operatorname{div} b) \cdot b + (a \bmod b)$

If $d \mid a$ and $d \mid b$ then $d \mid (a \bmod b)$.

If $d \mid b$ and $d \mid (a \bmod b)$ then $d \mid a$.

Euclid's Algorithm

GCD(x, y) = GCD(y, x mod y)

```
int GCD(int a, int b){ /* a >= b, b > 0 */
    int tmp;
    while (b > 0) {
        tmp = a % b;
        a = b;
        b = tmp;
    }
    return a;
}
```

Example: GCD(660, 126)

Goal: Solve $ax \equiv b \pmod{m}$ for unknown x.

Idea: Find a number z such that $za \equiv 1 \pmod{m}$.

Multiply both sides by z:

$$ax \equiv b \pmod{m}$$
$$zax \equiv zb \pmod{m}$$
$$x \equiv zb \pmod{m}$$

If such an element exists, we use the notation a^{-1} so that

$$a^{-1}a \equiv aa^{-1} \equiv 1 \pmod{m}$$

a^{-1} is called the **multiplicative inverse of a modulo m.**

When is there an inverse?

Theorem: a has a multiplicative inverse modulo m **if and only if** $\gcd(a, m) = 1$.

Bezout's Theorem

If *a* and *b* are positive integers, then there exist integers ***s*** and ***t*** such that

$$\gcd(a,b) = sa + tb$$

For example: $1 = \gcd(27, 35) = 13 \cdot 27 + (-10) \cdot 35$

If $\gcd(a, m) = 1$ then we can write

$$1 = \gcd(a, m) = sa + tm$$

for some integers s, t.

So $sa \equiv 1 \pmod{m}$.

Thus $a^{-1} = s$ is the inverse!

extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that $\gcd(a, b) = sa + tb$
- e.g. gcd(35,27):

35 = 1 • 27 + 8	35 - 1 • 27 = 8
27= 3 • 8 + 3	27- 3 • 8 = 3
8 = 2 • 3 + 2	8 - 2 • 3 = 2
3 = 1 • 2 + 1	3 - 1 • 2 = 1
2 = 2 • 1 + 0	

- Substitute back from the bottom

1 = 3 - 1 • 2 = 3 – 1 (8 - 2 • 3) = (-1) • 8 + 3 • 3
= (-1) • 8 + 3 (27- 3 • 8) = 3 • 27 + (-10) • 8
= 3 • 27 + (-10) • (35 - 1 • 27) = -10 • 35 + 13 • 27

Solving $ax \equiv b \pmod{m}$ for unknown x when $\gcd(a, m) = 1$.

1. Find s such that $sa + tm = 1$
2. Compute $a^{-1} = s \bmod m$, the multiplicative inverse of a modulo m
3. Set $x = (a^{-1} \cdot b) \bmod m$

Solve: $7x \equiv 1 \pmod{26}$

For a multiplicative cipher to be **invertible**:

$$f : \{0, \ldots, m-1\} \to \{0, \ldots, m-1\}$$

$$f(x) = ax \bmod m$$

must be one-to-one and onto.

Lemma: If there is an integer b such that $ab \bmod m = 1$, then the function $f(x) = ax \bmod m$ is one-to-one and onto.

could we prove this?

If *a* and *b* are positive integers, then there exist integers ***s*** and ***t*** such that

$$\gcd(a,b) = \boldsymbol{s}a + \boldsymbol{t}b$$

Need a **new inference rule**.

Method for proving statements about all integers ≥ 0

- A new logical inference rule!
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- Particularly useful for reasoning about programs!

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
        else { return f(x-1)+1; }}
```

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

prove: for all $n > 0$, a is odd $\rightarrow a^n$ is odd

Let $n > 0$ be arbitrary.

Suppose that a is odd. We know that if a, b are odd, then ab is also odd.

So: $\left(\cdots \cdot \left((a \cdot a) \cdot a\right) \cdot \cdots \cdot a\right) = a^n$ [n times]

Those "$\cdots$"s are a problem! We're trying to say "we can use the same argument over and over…"

We'll come back to this.

induction is a rule of inference

Domain: Natural Numbers

$$
\begin{array}{l}
P(0) \\
\forall\, k\ (P(k) \ \rightarrow\ P(k+1)) \\
\hline
\therefore\ \forall\, n\ P(n)
\end{array}
$$

using the induction rule in a formal proof

$$P(0)$$
$$\forall\, k\, (P(k) \;\rightarrow\; P(k+1))$$

$$\therefore\ \forall\, n\, P(n)$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

format of an induction proof

$$P(0)$$
$$\forall\, k\, (P(k) \rightarrow P(k+1))$$

$$\therefore\ \forall\, n\, P(n)$$

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true

Inductive Hypothesis

    4. ...
    5. Prove P(k+1) is true

Inductive Step

6.	$P(k) \rightarrow P(k+1)$	Direct Proof Rule
7.	$\forall\, k\, (P(k) \rightarrow P(k+1))$	Intro ∀ from 2-6
8.	$\forall\, n\, P(n)$	Induction Rule 1&7

Conclusion

inductive proof in five easy steps

Proof:

1. "We will show that P(n) is true for every n ≥ 0 by **induction**."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !)
5. "Conclusion: Result follows by induction."

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

Can we describe the pattern?

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$$

proving $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

- We could try proving it normally…
  - We want to show that $1 + 2 + 4 + \cdots + 2^n = 2^{n+1}$.
  - So, what do we do now? We can sort of explain the pattern, but that's not a proof…
- We could prove it for n=1, n=2, n=3, … (individually), but that would literally take forever…

inductive proof in five easy steps

Proof:

1. "We will show that P(n) is true for every $n \geq 0$ by **induction**."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

   Assume P(k) is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Want to prove that P(k+1) is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !)
5. "Conclusion: Result follows by induction."

proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

1. Let P(n) be "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". We will show P(n) is true for all natural numbers by induction.
2. Base Case (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$
3. Induction Hypothesis: Suppose that P(k) is true for some arbitrary $k \geq 0$.
4. Induction Step:

   Goal: Show P(k+1), i.e. show $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

   $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ by IH

   Adding 2^{k+1} to both sides, we get:

   $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly P(k+1).
5. Thus P(k) is true for all $k \in \mathbb{N}$, by induction.

another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- $\ldots$

prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

For all $n \geq 1$: $1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$