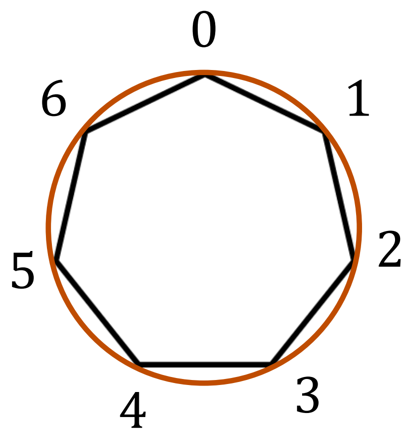
Fall 2015

Lecture 11: Modular arithmetic and applications

arithmetic mod 7

$$a +_7 b = (a + b) \bmod 7$$

$$a \times_7 b = (a \times b) \bmod 7$$



+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

review: division theorem

Let *a* be an integer and *d* a positive integer.
Then there are *unique* integers *q* and *r*, with
$0 \le r < d$, such that $a = d\,q + r$.

$q = a$ **div** d $\boxed{r = a \textbf{ mod } d}$

$a = 23$ $q = 3$

$d = 7$ $r = 2$

$a = -20$ $q = -3$

$d = 7$ $r = 1$

Note: $r \geq 0$ even if $a < 0$.
Not quite the same as **`a % d`**.

Let a and b be integers, and m be a positive integer.
We say *a* is **congruent** to *b* **modulo** *m* if *m* divides *a* – *b*.
We use the notation a ≡ b (mod m) to indicate that a is congruent to b modulo m.

$$a \equiv b \pmod{m}$$

if $m \mid a - b$

$3 \mid 0$? yes

modular arithmetic: examples

$0 = 3 \cdot 0$ $\boxed{2 \mid 0}$ ✓

A ≡ 0 (mod 2) $2 \mid A-0 \iff 2 \mid A$

This statement is the same as saying "A is even"; so, any A that is even (including negative even numbers) will work.

1 ≡ 0 (mod 4) $1 \equiv 0 \pmod 1$ $1 \mid (1-0) \iff 1 \mid 1$ ✓

This statement is false. If we take it mod 1 instead, then the statement is true.

A ≡ -1 (mod 17) $A = 17x - 1$ for any integer x

If A = 17x – 1 = 17(x-1) + 16 for an integer x, then it works.

Note that (m – 1) mod m

= ((m mod m) + (-1 mod m)) mod m

= (0 + -1) mod m

= -1 mod m

Theorem: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

Proof:

$r \mid s \iff s = kr$ for some int. k

$(a+b) \bmod m = a \bmod m + b \bmod m$

congruence and residues

$(6+6) \bmod 7 = 5 \neq 6 \bmod 7 + 6 \bmod 7$

Theorem: Let a and b be integers, and let m be a positive integer. Then a ≡ b (mod m) if and only if a mod m = b mod m.

Proof: ⇒ Assume $a \equiv b \pmod{m}$

Suppose that a ≡ b (mod m).

Then $m \mid a-b$ by def. of $\equiv$.

By definition: a ≡ b (mod m) implies m | (a − b)

which by definition implies that a − b = km for some integer k.

So $a-b = km$ for some k

$a = km+b$

Therefore a = b + km.

$a \bmod m = (km+b) \bmod m$

Taking both sides modulo m we get

By div. thm $a = q m + r$ $\quad 0 \le r < m$

a mod m = (b+km) mod m = b mod m

$b = q' m + r'$ $\quad 0 \le r' < m$

$a-b = (q-q')m + (r-r')$

$\implies km = (q-q')m + (r-r')$

...

Theorem: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

Proof: $\Rightarrow$

Suppose that $a \equiv b \pmod{m}$.

By definition: $a \equiv b \pmod{m}$ implies $m \mid (a - b)$

which by definition implies that $a - b = km$ for some integer k.

Therefore $a = b + km$.

Taking both sides modulo m we get

$a \bmod m = (b+km) \bmod m = b \bmod m$

Div thm: $b + km = qm + r \quad 0 \le r < m$

$\Longrightarrow \quad b = (q-k)m + r \quad 0 \le r < m$

$\Longrightarrow \quad b \bmod m = r = (b+km) \bmod m$

Theorem: Let a and b be integers, and let m be a positive integer. Then a ≡ b (mod m) if and only if a mod m = b mod m.

Proof: Assume $a \bmod m = b \bmod m$

$\Rightarrow \quad a = qm + r \qquad 0 \le r < m$

$b = q'm + r \qquad 0 \le r < m$

$\Rightarrow \quad a - b = (q - q')m + (r - r)$

$= (q - q')m$

$\Rightarrow \quad m \mid a - b \quad \Rightarrow \quad a \equiv b \pmod{m}$

(by def.)

Theorem: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if a mod m = b mod m.

Proof: $\Leftarrow$

Suppose that a mod m = b mod m.

By the division theorem, a = mq + (a mod m) and
b = ms + (b mod m) for some integers q,s.

$$
\begin{aligned}
a - b &= (mq + (a \bmod m)) - (mr + (b \bmod m)) \\
&= m(q - r) + (a \bmod m - b \bmod m) \\
&= m(q - r) \quad \text{since } a \bmod m = b \bmod m
\end{aligned}
$$

Therefore m | (a-b) and so $a \equiv b \pmod{m}$

consistency of addition

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{a + c \equiv b + d \pmod{m}}$

consistency of addition

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{a + c \equiv b + d \pmod{m}}$

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

Unrolling definitions gives us some k such that $a - b = km$, and some j such that $c - d = jm$.

Adding the equations together gives us $(a + c) - (b + d) = m(k + j)$. Now, re-applying the definition of mod gives us $a + c \equiv b + d \pmod{m}$.

$(k\cancel{m}+b)(j\cancel{m}+d) \equiv bd \pmod{m}$

consistency of multiplication

Let m be a positive integer. If a ≡ b (mod m) and c ≡ d (mod m), then **ac ≡ bd (mod m)**

$3^6 \equiv (-2)^6 \equiv 4 \pmod{5}$

~~$3^6 \equiv 3^1 \pmod{5}$ $\equiv 3 \pmod{5}$~~

Suppose a ≡ b (mod m) and c ≡ d (mod m).

Unrolling definitions gives us some k such that a – b = km, and some j such that c – d = jm.

Then, a = km + b and c = jm + d.

$ac = bd + m(kjm + dk + bj)$

Multiplying both together gives us

ac = (km + b)(jm + d) = kjm^2 + kmd + jmb + bd

Rearranging gives us ac – bd = m(kjm + kd + jb).

Using the definition of mod gives us ac ≡ bd (mod m).

Let n be an integer.
Prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

n even $\Rightarrow n = 2k$ for some k

$\Rightarrow n^2 = 4k^2 \Rightarrow n^2 \equiv 0 \pmod 4$

$0^2 \equiv 0 \pmod 4$

$1^2 \equiv 1 \pmod 4$

$2^2 \equiv 4 \equiv 0 \pmod 4$

$3^2 \equiv 9 \equiv 1 \pmod 4$

$n^2 \equiv (\underbrace{n \bmod 4}_{0,1,2,3})^2 \pmod 4$

Let n be an integer.
Prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

Case 1 (n is even):

Suppose $n \equiv 0 \pmod 2$.
Then, $n = 2k$ for some integer k.
So, $n^2 = (2k)^2 = 4k^2$.
So, by definition of congruence, $n^2 \equiv 0 \pmod 4$.

Case 2 (n is odd):

Suppose $n \equiv 1 \pmod 2$.
Then, $n = 2k + 1$ for some integer k.
So, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$.
So, by definition of congruence, $n^2 \equiv 1 \pmod 4$.

n-bit unsigned integer representation

- Represent integer x as sum of powers of 2:
  If $x = \sum_{i=0}^{n-1} b_i 2^i$ where each $b_i \in \{0,1\}$
  then representation is $b_{n-1} \cdots b_2\, b_1\, b_0$

  99 = 64 + 32 + 2 + 1
  18 = 16 + 2

- For n = 8:

  99: 0110 0011
  18: 0001 0010

sign-magnitude integer representation

n-bit signed integers

Suppose $-2^{n-1} < x < 2^{n-1}$

First bit as the sign, n-1 bits for the value

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

99: 0110 0011

-18: 1001 0010

Any problems with this representation?

1

0 11

+ 0 11

-2 $\boxed{110}$

3

3

6

$\boxed{\begin{array}{l} 0000\ 0000 \\ 1000\ 0000 \end{array}}$

Yes.

two's complement representation

n-bit signed integers, first bit will still be the sign bit

Suppose $0 \leq x < 2^{n-1}$,

x is represented by the binary representation of x

Suppose $0 \leq x \leq 2^{n-1}$,

$-x$ is represented by the binary representation $\boxed{\text{of } 2^n - x}$

Key property: Two's complement representation of any number y is equivalent to y mod 2^n so arithmetic works mod 2^n

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

99: 0110 0011

-18: 1110 1110

sign-magnitude vs. two's complement

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1111	1110	1101	1100	1011	1010	1001	0000	0001	0010	0011	0100	0101	0110	0111

Sign-Magnitude

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111

Two's complement

two's complement representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $2^n - x$

- To compute this: Flip the bits of x then add 1:
  - All 1's string is $2^n - 1$, so

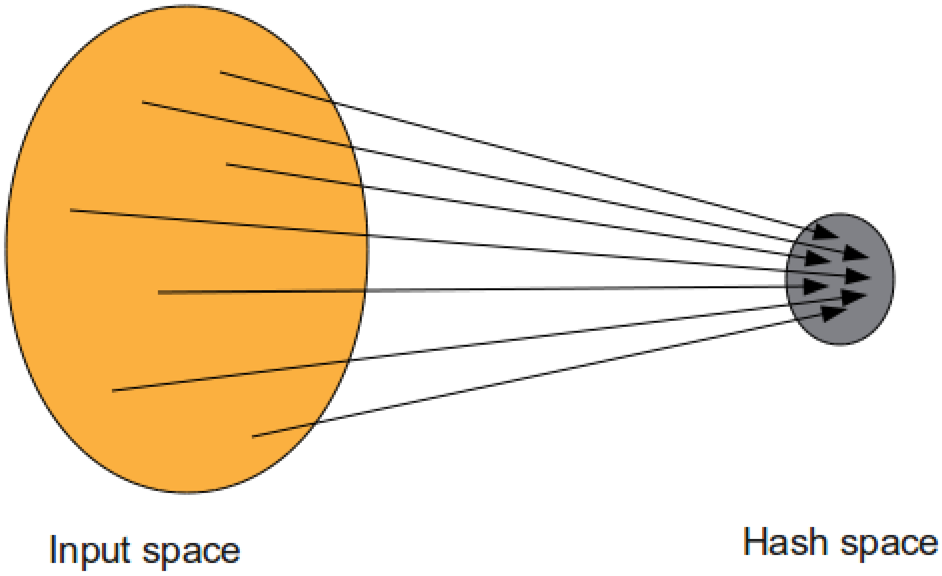
    Flip the bits of x $\equiv$ replace x by $2^n - 1 - x$

basic applications of mod

- Hashing
- Pseudo random number generation
- Simple cipher

Scenario:

Map a small number of data values from a large domain $\{0, 1, \ldots, M - 1\}$ into a small set of locations $\{0,1, \ldots, n - 1\}$ so one can quickly check if some value is present.



Scenario:

Map a small number of data values from a large domain $\{0, 1, \ldots, M-1\}$ into a small set of locations $\{0,1, \ldots, n-1\}$ so one can quickly check if some value is present

- $\mathrm{hash}(x) = x \bmod p$ for p a prime close to n
  - or $\mathrm{hash}(x) = (ax + b) \bmod p$

- Depends on all of the bits of the data
  - helps avoid collisions due to similar values
  - need to manage them if they occur

pseudo-random number generation

Linear Congruential method:

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random x_0, a, c, m and produce a long sequence of x_n's

[good for some applications, really bad for many others

simple ciphers

- Caesar cipher, A = 1, B = 2, . . .
  - HELLO WORLD
- Shift cipher
  - $f(p) = (p + k) \bmod 26$
  - $f^{-1}(p) = (p - k) \bmod 26$
- More general
  - $f^{-1}(p) = (ap + b) \bmod 26$

modular exponentiation mod 7

x	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

modular exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

modular exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1	1	1	1	1	1	1
2	2	4	1	2	4	1
3	3	2	6	4	5	1
4	4	2	1	4	2	1
5	5	4	6	2	3	1
6	6	1	6	1	6	1