cse 311: foundations of computing

Fall 2015

Lecture 11: Modular arithmetic and applications

Domain Pos Real

$x^2 \to x$

~~one-to-one~~ x

~~onto~~ x

one-to-one

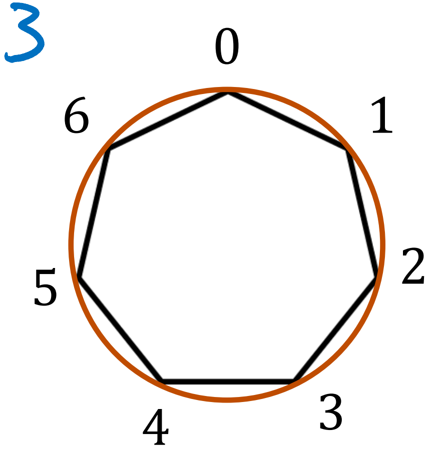
$f: A \to B$

arithmetic mod 7

$6 \times_7 4 = 3$

$a +_7 b = (a + b) \bmod 7$

$a \times_7 b = (a \times b) \bmod 7$



+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Let a be an integer and d a positive integer. Then there are *unique* integers q and r, with $0 \le r < d$, such that $a = d\,q + r$.

$q = a$ **div** d $\qquad\qquad$ $r = a$ **mod** d

$a = -13 \qquad d = 4$

$q = -4 \qquad r = 3$

$a = dq \iff d \mid a$

Note: r ≥ 0 even if a < 0.
Not quite the same as **`a % d`**.

Let a and b be integers, and m be a positive integer.
We say *a* is **congruent** to *b* **modulo** *m* if *m* divides *a* – *b*.
We use the notation a ≡ b (mod m) to indicate that a is congruent to b modulo m.

$a \equiv b \mod m \iff m \mid a-b$

$\Updownarrow$

$\exists k \quad a-b = km$

$b-a = -km$

$m \mid a-b \iff m \mid b-a$

$10 \equiv 20 \mod 5$

$10 \equiv 21 \mod 1$

$21 \equiv 10 \mod 1$

modular arithmetic: examples

$a \equiv b \mod m \iff a \equiv b+m \mod m$

$\iff a+m \equiv b+m \mod m$

$A \equiv 0 \pmod 2$

This statement is the same as saying "A is even"; so, any A that is even (including negative even numbers) will work.

$\cancel{1 \equiv 0 \pmod 4}$

This statement is false. If we take it mod 1 instead, then the statement is true.

$A \equiv -1 \pmod{17} \equiv 16 \mod 17$

If A = 17x – 1 = 17(x-1) + 16 for an integer x, then it works.

Note that (m – 1) mod m

= ((m mod m) + (-1 mod m)) mod m

= (0 + -1) mod m

= -1 mod m

Theorem: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod m$ if and only if $a \bmod m = b \bmod m$.

Proof:

$a \equiv b \bmod m \iff a \bmod m = b \bmod m$

$\Rightarrow$ (only if) direction

$m \mid a-b$ def cong

$\exists k\ a-b = m.k$ def div

$a = b + mk$ $\quad m \cdot b \operatorname{div} m + b \bmod m$

$a \bmod m = (b + mk) \bmod m = b \bmod m + \underbrace{mk \bmod m}_{0}$

$= b \bmod m$.

(

Theorem: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

Proof: $\Rightarrow$

Suppose that $a \equiv b \pmod{m}$.

By definition: $a \equiv b \pmod{m}$ implies $m \mid (a - b)$

which by definition implies that $a - b = km$ for some integer k.

Therefore $a = b + km$.

Taking both sides modulo m we get

$a \bmod m = (b+km) \bmod m = b \bmod m$

Theorem: Let a and b be integers, and let m be a positive integer. Then a ≡ b (mod m) if and only if a mod m = b mod m.

Proof:

$a \bmod m = b \bmod m \Rightarrow a \equiv b \bmod m.$

$a = (a \text{ div } m) \cdot m + a \bmod m$

$b = (b \text{ div } m) \cdot m + b \bmod m$

$a - b = m\underbrace{(a \text{ div } m - b \text{ div } m)}_{\text{int}} + \underbrace{a \bmod m - b \bmod m}_{0}$

$\exists k \quad a - b = mk$

$m \mid a - b$

$a \equiv b \bmod m.$

Theorem: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if a mod m = b mod m.

Proof: $\Leftarrow$

Suppose that a mod m = b mod m.

By the division theorem, a = mq + (a mod m) and
b = ms + (b mod m) for some integers q,s.

$$\begin{aligned} a - b &= (mq + (a \bmod m)) - (mr + (b \bmod m)) \\ &= m(q - r) + (a \bmod m - b \bmod m) \\ &= m(q - r) \quad \text{since } a \bmod m = b \bmod m \end{aligned}$$

Therefore m | (a-b) and so $a \equiv b \pmod{m}$

consistency of addition

Let m be a positive integer. If a ≡ b (mod m) and c ≡ d (mod m), then **a + c ≡ b + d (mod m)**

$a \equiv b \mod m \iff m \mid a-b \iff a-b = m.k$

$c \equiv d \mod m \iff m \mid c-d \iff c-d = m.j$

$a-b+c-d = m(j+k)$

$m \mid a+c-(b+d)$

$a+c \equiv b+d \mod m.$

consistency of addition

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{a + c \equiv b + d \pmod{m}}$

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

Unrolling definitions gives us some k such that $a - b = km$, and some j such that $c - d = jm$.

Adding the equations together gives us $(a + c) - (b + d) = m(k + j)$. Now, re-applying the definition of mod gives us $a + c \equiv b + d \pmod{m}$.

consistency of multiplication

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then **$ac \equiv bd \pmod{m}$**

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
Unrolling definitions gives us some k such that $a - b = km$, and some j such that $c - d = jm$.

Then, $a = km + b$ and $c = jm + d$.
Multiplying both together gives us

$$ac = (km + b)(jm + d) = kjm^2 + kmd + jmb + bd$$

mult of m

Rearranging gives us $ac - bd = m(kjm + kd + jb)$.
Using the definition of mod gives us $ac \equiv bd \pmod{m}$.

Let n be an integer.
Prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

Pf $n \equiv 0 \mod 2$

$n = 2k \quad n^2 = 4k^2 \quad 4 \mid n^2 \quad n^2 \equiv 0 \mod 4.$

$n \equiv 1 \mod 2$

$n = 2k+1 \quad n^2 = 4k^2 + 4k + 1 = 4(k^2+k)+1$

$n^2 \equiv 1 \mod 4$

Pf 2. $n \equiv 0 \mod 4 \Rightarrow n^2 \equiv 0^2 \mod 4$

$n \equiv 1 \mod 4 \Rightarrow n^2 \equiv 1^2 = 1 \mod 4$

$n \equiv 2 \mod 4 \Rightarrow n^2 \equiv 2^2 \equiv 0 \mod 4$

$n \equiv 3 \mod 4 \Rightarrow n^2 \equiv 3^2 \equiv 1 \mod 4.$

example

Let n be an integer.
Prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

Case 1 (n is even):

Suppose $n \equiv 0 \pmod 2$.
Then, $n = 2k$ for some integer k.
So, $n^2 = (2k)^2 = 4k^2$.
So, by definition of congruence, $n^2 \equiv 0 \pmod 4$.

Case 2 (n is odd):

Suppose $n \equiv 1 \pmod 2$.
Then, $n = 2k + 1$ for some integer k.
So, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$.
So, by definition of congruence, $n^2 \equiv 1 \pmod 4$.

n-bit unsigned integer representation

- Represent integer x as sum of powers of 2:
  If $x = \sum_{i=0}^{n-1} b_i 2^i$ where each $b_i \in \{0,1\}$
  then representation is $b_{n-1} \cdots b_2\, b_1\, b_0$

  99 = 64 + 32 + 2 + 1
  18 = 16 + 2

- For n = 8:

  99: 0110 0011
  18: 0001 0010

sign-magnitude integer representation

n-bit signed integers

Suppose $-2^{n-1} < x < 2^{n-1}$

First bit as the sign, n-1 bits for the value

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

99: 0110 0011

-18: 1001 0010

Any problems with this representation?

two's complement representation

n-bit signed integers, first bit will still be the sign bit

Suppose $0 \leq x < 2^{n-1}$,

x is represented by the binary representation of x

Suppose $0 \leq x \leq 2^{n-1}$,

$-x$ is represented by the binary representation of $2^n - x$

Key property: Two's complement representation of any number y is equivalent to y mod 2^n so arithmetic works mod 2^n

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

99: 0110 0011

-18: 1110 1110

sign-magnitude vs. two's complement

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1111	1110	1101	1100	1011	1010	1001	0000	0001	0010	0011	0100	0101	0110	0111

Sign-Magnitude

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111

Two's complement

two's complement representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $2^n - x$

- To compute this: Flip the bits of x then add 1:
  - All 1's string is $2^n - 1$, so

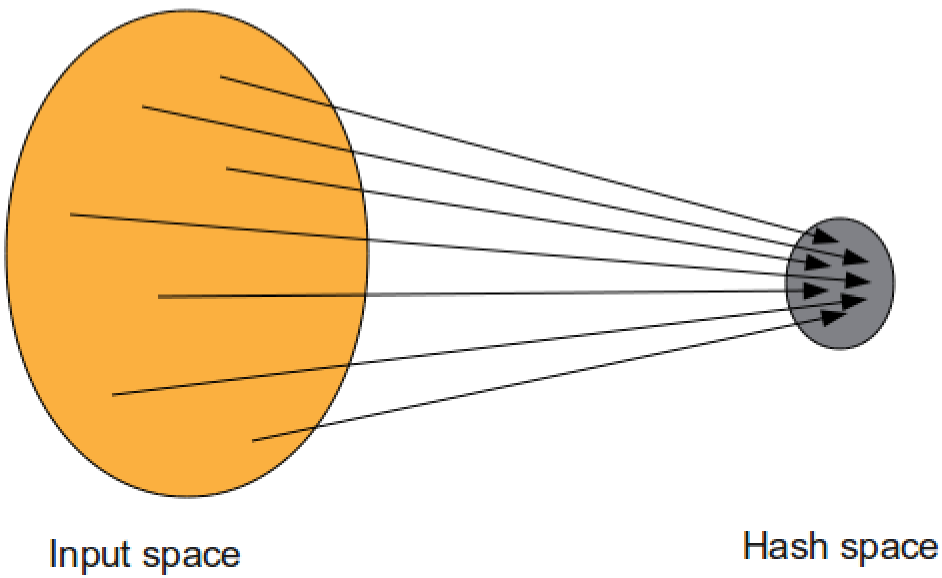
    Flip the bits of x $\equiv$ replace x by $2^n - 1 - x$

basic applications of mod

- Hashing
- Pseudo random number generation
- Simple cipher

Scenario:

Map a small number of data values from a large domain $\{0, 1, \dots, M-1\}$ into a small set of locations $\{0, 1, \dots, n-1\}$ so one can quickly check if some value is present.



Scenario:

Map a small number of data values from a large domain $\{0, 1, \ldots, M-1\}$ into a small set of locations $\{0,1, \ldots, n-1\}$ so one can quickly check if some value is present

- $\text{hash}(x) = x \bmod p$ for p a prime close to n
  - or $\text{hash}(x) = (ax + b) \bmod p$

- Depends on all of the bits of the data
  - helps avoid collisions due to similar values
  - need to manage them if they occur

Linear Congruential method:

$$x_{n+1} = (a\,x_n + c) \bmod m$$

Choose random x_0, a, c, m and produce a long sequence of x_n's

[good for some applications, really bad for many others

simple ciphers

- Caesar cipher, A = 1, B = 2, . . .
  - HELLO WORLD
- Shift cipher
  - f(p) = (p + k) mod 26
  - f^{-1}(p) = (p – k) mod 26
- More general
  - f^{-1}(p) = (ap + b) mod 26

modular exponentiation mod 7

x	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

modular exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

modular exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1	1	1	1	1	1	1
2	2	4	1	2	4	1
3	3	2	6	4	5	1
4	4	2	1	4	2	1
5	5	4	6	2	3	1
6	6	1	6	1	6	1