Power set of a set $A =$ set of all subsets of A

$$\boxed{\mathcal{P}(A) = \{\, B : B \subseteq A \,\}}$$

$A \subseteq A$

e.g. $\text{Days} = \{M, W, F\}$

$$\mathcal{P}(\text{Days}) = \{ \varnothing,\ \{M\}, \{W\}, \{F\},\ \{M,W\}, \{W,F\}, \{M,F\},\ \{M,W,F\} \ \}$$

e.g. $\mathcal{P}(\varnothing) = \{\varnothing\} \neq \varnothing$

$\varnothing \subseteq \varnothing$

$\{\mathcal{P}(A)\} \subseteq \mathcal{P}(\mathcal{P}(A))$

review quiz

$\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$

false

$\mathcal{P}(A) \in \mathcal{P}(\mathcal{P}(A))$

true

Since $B \subseteq B$,

$B \in \mathcal{P}(B)$

$B = \mathcal{P}(A)$

$|A| = k$

If $|A| = n$, what is $|\mathcal{P}(A)|$?

$|\mathcal{P}(A)| = 2^k$

$|\mathcal{P}(\mathcal{P}(A))| = 2^{(2^k)}$

size of $|\underbrace{A \times A \times A \times A}_{n^4}|$?

$|A \times A| = n^2$

$A = \{1\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}\}$

$\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$

Fall 2015

Lecture 10: Functions, Modular arithmetic

a little recap

So far:

- Propositional logic
- Logic to build circuits
- Predicates and quantifiers
- Proof systems and logical inference
- Basic set theory

Question: If the domain of discourse is empty and P is a predicate, what is the truth value of:

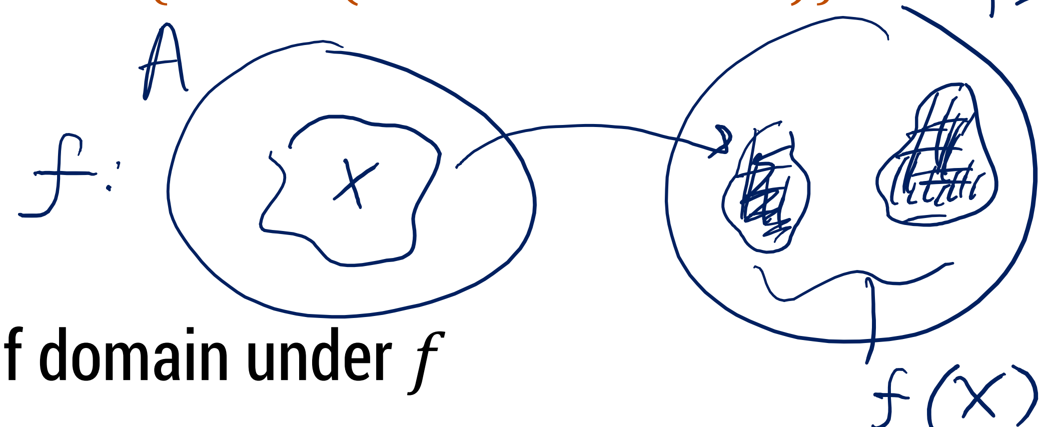
$\exists x\ P(x)$ F

$\forall x\ P(x)$ T

A **function** from A to B:

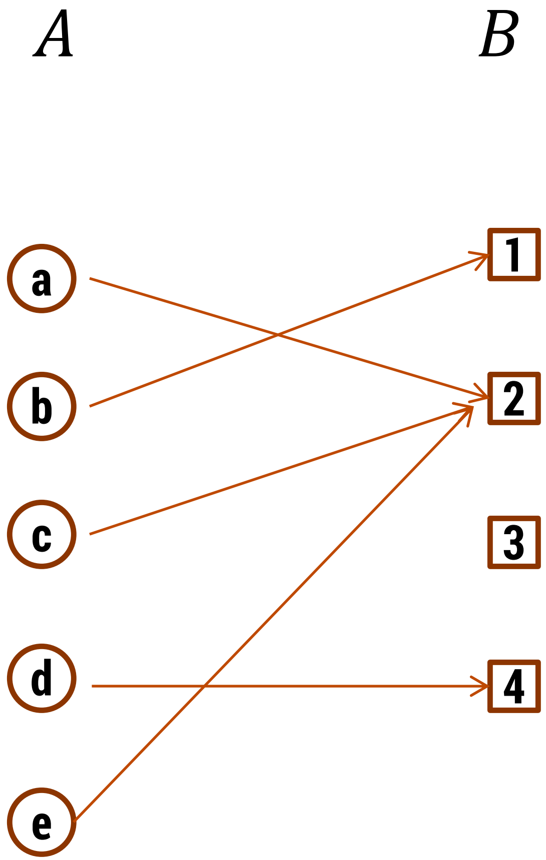
- Every element of A is assigned to exactly one element of B.
- We write $f : A \to B$.
- "Image of X under f" $=$ "$f(X)$"

$$= \{x : \exists y\,(y \in X \wedge x = f(y))\}$$



- **Domain** of f is A
- **Codomain** of f is B
- **Image** of f = Image of domain under f
  = all the elements pointed to by something in the domain.

image

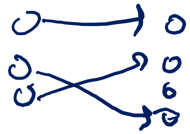


Image({a}) = {2}

Image({a, e}) = {2}

Image({a, b}) = {1, 2}

Image(A) = {1, 2, 4}

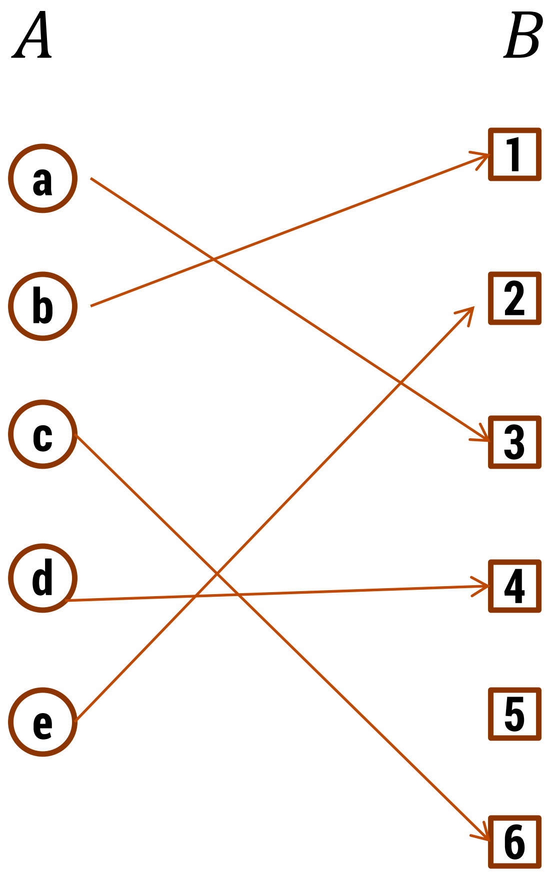


injections and surjections

A **function** $f : A \to B$ is **one-to-one** (or, **injective)** if every output corresponds to at most one input, i.e. $f(x) = f(x') \Rightarrow x = x'$ for all $x, x' \in A$.

A **function** $f : A \to B$ is **onto** (or, **surjective)** if every output gets hit, i.e. for every $y \in B$, there exists $x \in A$ such that $f(x) = y$.

is this function one-to-one? is it onto?



injective

one-to-one ✓

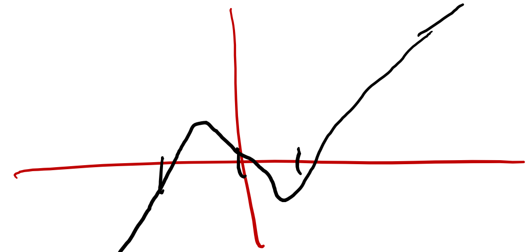
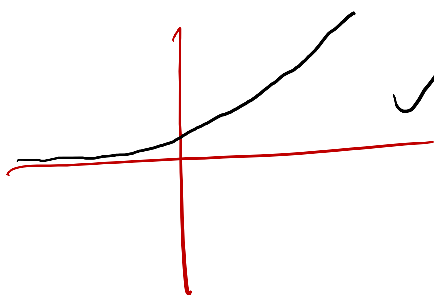
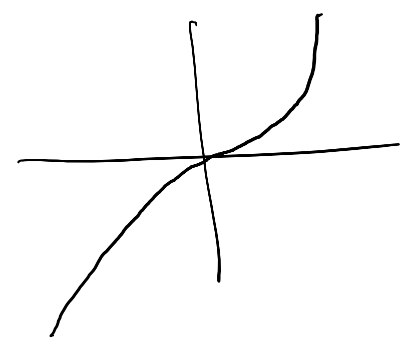
It is one-~~to-one~~, because nothing in B is pointed to by multiple elements of A.

onto

surjective

It is not onto, because 5 is not pointed to by anything.

QUIZ!

	One-to-one (?)		Onto (?)	
$x \mapsto x^2$	$-2 \longmapsto 4$ $2 \longmapsto 4$	✗	doesn't hit neg numbers	✗
$x \mapsto x^3 - x$	$1 \longmapsto 0$ $0 \longmapsto 0$	✗		✓
$x \mapsto e^x$		✓		✗
$x \mapsto x^3$	$x \longmapsto \lvert x \rvert^{1/3} \operatorname{sign}(x)$ ✓			✓

Co domain
Domain: Reals ''

a harder quiz

“number theory” (and applications to computing)

- How whole numbers work

  [fascinating, deep, weird area of mathematics that no one understands,
  but the basics are easy and really useful]

- Many significant applications
  - Cryptography [this is how SSL works]
  - Hashing
  - Security
  - Error-correcting codes [this is how your bluray player works]

- Important tool set

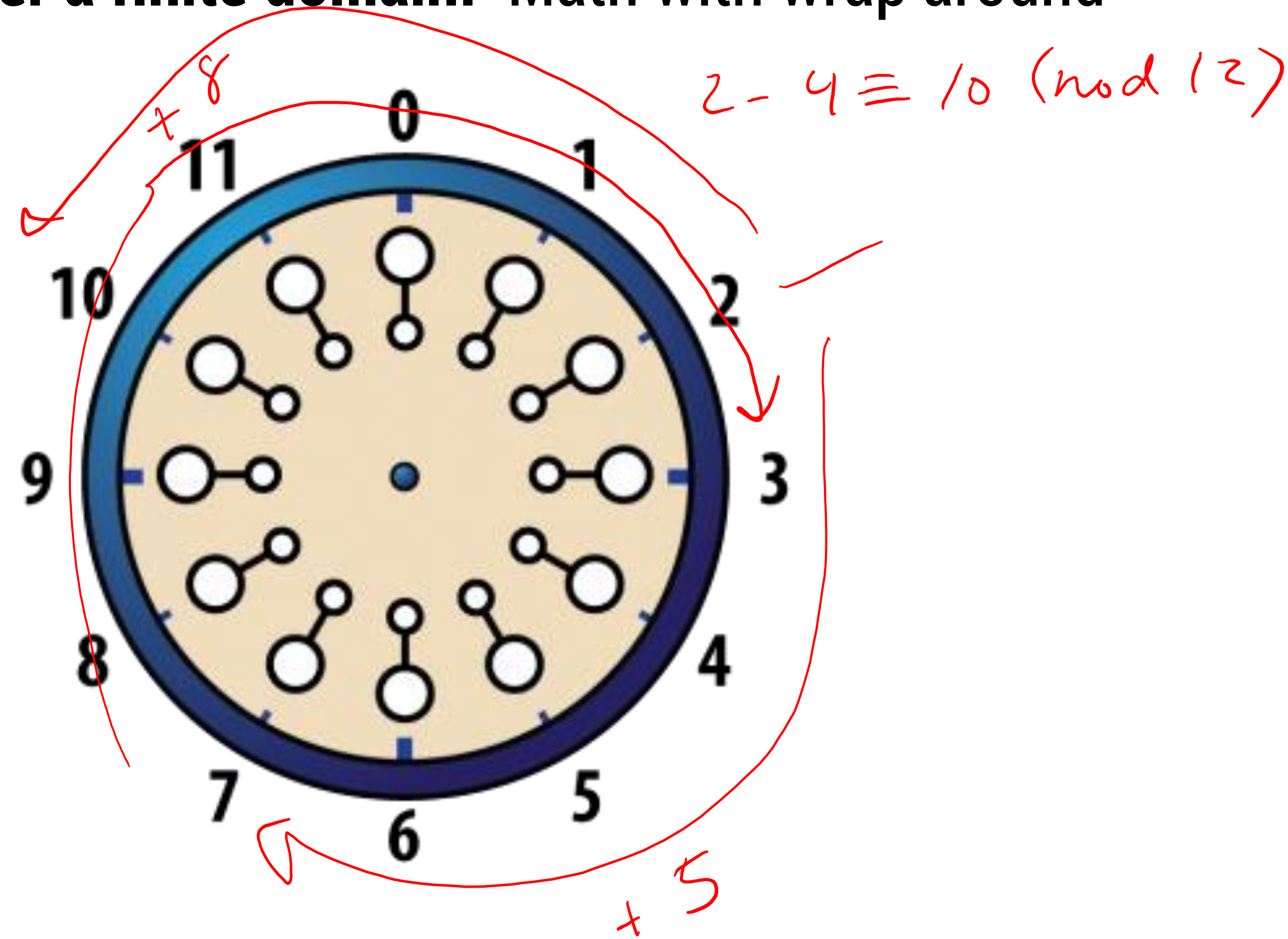
thanks, java

```
public class Test {
    final static int SEC_IN_YEAR = 364*24*60*60;
    public static void main(String args[]) {
        System.out.println(
            “I will be alive for at least ” +
            SEC_IN_YEAR * 101 + “ seconds.”
        );
    }
}
```

```
 ----jGRASP exec: java Test
I will be alive for at least -186619904 seconds.

 ----jGRASP: operation complete.
```

Arithmetic over a finite domain: Math with wrap around



Integers a, b, with a ≠ 0. We say that a **divides** b iff there is an integer k such that b = k a. The notation a | b denotes "a divides b."

$3 \mid 15$

$1 \mid 15$

$15 \nmid 17$

$\overset{a}{3} \mid \overset{b}{0}$?

$0 = 0 \cdot 3$

$b = k \; a$

division theorem

Let *a* be an integer and *d* a positive integer. Then there are *unique* integers *q* and *r*, with $0 \le r < d$, such that $a = d\,q + r$.

$q = a$ **div** d $\qquad\qquad r = a$ **mod** d

$a = 15$ $\qquad q = 3$

$d = 4$ $\qquad r = 3$

$a = -13$ $\qquad q = -4$

$d = 4$ $\qquad r = 3$

Note: r ≥ 0 even if a < 0.
Not quite the same as **`a % d`**.

arithmetic mod 7

$$a +_7 b = (a + b) \bmod 7$$

$$a \times_7 b = (a \times b) \bmod 7$$

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

							−2	−1
	x	0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
	1	0	1	2	3	4	5	6
	2	0	2	4	6	1	3	5
	3	0	3	6	2	5	1	4
	4	0	4	1	5	2	6	3
−2	5	0	5	3	1	6	4	2
−1	6	0	6	5	4	3	2	1

modular congruence

Let a and b be integers, and m be a positive integer.
We say *a* is **congruent** to *b* **modulo** *m* if *m* divides *a* – *b*.
We use the notation a ≡ b (mod m) to indicate that a is congruent to b modulo m.

$$a \equiv b \pmod{m}$$

$$\iff \quad m \mid a-b$$

modular arithmetic: examples

$A \equiv 0 \pmod 2$

This statement is the same as saying "A is even"; so, any A that is even (including negative even numbers) will work.

$1 \equiv 0 \pmod 4$

This statement is false. If we take it mod 1 instead, then the statement is true.

$A \equiv -1 \pmod{17}$

If $A = 17x - 1 = 17(x-1) + 16$ for an integer x, then it works.

Note that $(m - 1) \bmod m$

$$= ((m \bmod m) + (-1 \bmod m)) \bmod m$$
$$= (0 + -1) \bmod m$$
$$= -1 \bmod m$$

modular arithmetic can haz sense

Theorem: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

Proof: Suppose that $a \equiv b \pmod{m}$.

By definition: $a \equiv b \pmod{m}$ implies $m \mid (a - b)$

which by definition implies that $a - b = km$ for some integer k.

Therefore $a = b + km$.

Taking both sides modulo m we get

$a \bmod m = (b+km) \bmod m = b \bmod m$

modular arithmetic can haz sense

Theorem: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

Proof: Suppose that $a \bmod m = b \bmod m$.

By the division theorem, $a = mq + (a \bmod m)$ and
$b = ms + (b \bmod m)$ for some integers q,s.

$$
\begin{aligned}
a - b &= (mq + (a \bmod m)) - (mr + (b \bmod m)) \\
&= m(q - r) + (a \bmod m - b \bmod m) \\
&= m(q - r) \quad \text{since } a \bmod m = b \bmod m
\end{aligned}
$$

Therefore $m \mid (a-b)$ and so $a \equiv b \pmod{m}$

consistency of addition

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{a + c \equiv b + d \pmod{m}}$

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

Unrolling definitions gives us some k such that $a - b = km$, and some j such that $c - d = jm$.

Adding the equations together gives us $(a + c) - (b + d) = m(k + j)$. Now, re-applying the definition of mod gives us $a + c \equiv b + d \pmod{m}$.

consistency of multiplication

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{ac \equiv bd \pmod{m}}$

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.

Unrolling definitions gives us some k such that $a - b = km$, and some j such that $c - d = jm$.

Then, $a = km + b$ and $c = jm + d$.

Multiplying both together gives us

$ac = (km + b)(jm + d) = kjm^2 + kmd + jmb + bd$

Rearranging gives us $ac - bd = m(kjm + kd + jb)$.

Using the definition of mod gives us $ac \equiv bd \pmod{m}$.

Let n be an integer.
Prove that $n^2 \equiv 0 \pmod{4}$ or $n^2 \equiv 1 \pmod{4}$