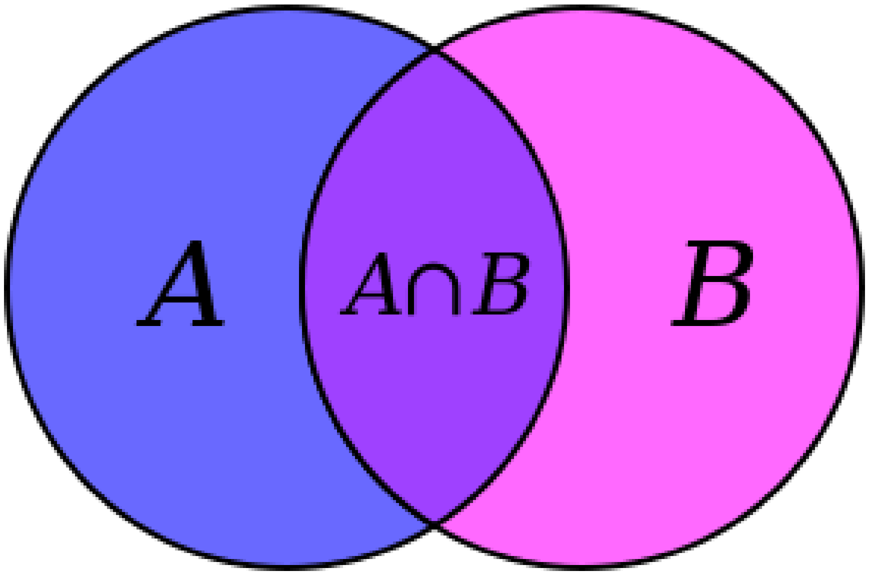
Fall 2015

Lecture 9: Set theory

- Formal treatment dates from late 19th century
- Direct ties between set theory and logic
- Important foundational language



some common sets

$\mathbb{N}$ is the set of **Natural numbers;** $\mathbb{N}$ = {0, 1, 2, …}
$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z}$ = {…, -2, -1, 0, 1, 2, …}
$\mathbb{Q}$ is the set of **Rational numbers**; e.g. ½, -17, 32/48
$\mathbb{R}$ is the set of **Real numbers**; e.g. 1, -17, 32/48, π
[n] is the set {1, 2, …, n} when n is a natural number
{} = ∅ is the **empty set**; the *only* set with no elements

EXAMPLES
Are these sets?
A = {1, 1}
B = {1, 3, 2}
C = {□, 1}
D = {{}, 17}
E = {1, 2, 7, cat, dog, ∅, α}

Set membership:
We write $2 \in E$; $3 \notin E$.

definitions

- A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\,(x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\,(x \in A \rightarrow x \in B)$$

$A = \{1, 2, 3\}$
$B = \{3, 4, 5\}$
$C = \{3, 4\}$

QUESTIONS

$\emptyset \subseteq A$? ✓
$A \subseteq B$? ✗
$C \subseteq B$ ✓

- A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\,(x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\,(x \in A \rightarrow x \in B)$$

- Note: $$(\boldsymbol{A} = \boldsymbol{B}) \equiv (\boldsymbol{A} \subseteq \boldsymbol{B}) \wedge (\boldsymbol{B} \subseteq \boldsymbol{A})$$

building sets from predicates

- The following says "S is the set of all x's where P(x) is true."

$$S = \{x : P(x)\}$$

- The following says "S is the set of those elements of A for which P(x) is true."

$$S = \{x \in A : P(x)\}$$

- "The set of all the real numbers less than one"

$$\{x \in \mathbb{R} : x < 1\}$$

- "The set of all powers of two"

$$\{x \in \mathbb{N} : \exists j\ (x = 2^j)\}$$

$$A \cup B = \{\, x : (x \in A) \vee (x \in B)\}$$ Union

$$A \cap B = \{\, x : (x \in A) \wedge (x \in B)\}$$ Intersection

$$A \setminus B = \{\, x : (x \in A) \wedge (x \notin B)\}$$ Set difference

A = {1, 2, 3}
B = {4, 5, 6}
C = {3, 4}

QUESTIONS

Using A, B, C and set operations, make…

[6] = ? $A \cup B$

{3} = ? $A \cap C$

{1,2} = ? $A \setminus C$

{1,3} = ? (Can we ever separate 1 and 2?)

more set operations

$$A \oplus B = \{\, x : (x \in A) \oplus (x \in B) \}$$

Symmetric difference

$$\overline{A} = \{\, x : x \notin A \,\}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 4, 2, 6}
C = {1, 2, 3, 4}

QUESTIONS

Let S = {1, 2}.

If the universe is A, then $\overline{S}$ is... {3}

If the universe is B, then $\overline{S}$ is... {4, 6}

If the universe is C, then $\overline{S}$ is... {3, 4}

it's Boolean algebra again! (yay…?)

- Definition for $\cup$ based on $\vee$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

- Definition for $\cap$ based on $\wedge$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

- Complement works like $\neg$

$$\overline{A} = \{x : \neg(x \in A)\}$$

Power set of a set A = set of all subsets of A

$$\mathcal{P}(A) = \{\, B : B \subseteq A \,\}$$

e.g. $\text{Days} = \{M, W, F\}$

$$\mathcal{P}(\text{Days}) = \{\, \emptyset, \{M\}, \{W\}, \{F\}, \{M, W\}, \{W, F\}, \{M, F\}, \{M, W, F\} \,\}$$

e.g. $\mathcal{P}(\emptyset) = ?$ $\quad (\mathcal{P}(\emptyset) = \{\emptyset\})$

cartesian product

$$A \times B = \{\, (a, b) : a \in A, b \in B \,\}$$

$A = \{1, 2\}$ $B = \{a, b, c\}$

$A \times B = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \}$

$A \times \emptyset = \emptyset$

$|A \times B| = |A| \cdot |B|$

de Morgan's laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$x \in \overline{A \cup B} \leftrightarrow \neg(x \in A \vee x \in B)$

$\leftrightarrow (\neg(x \in A) \wedge \neg(x \in B))$

$\leftrightarrow (x \in \bar{A} \wedge x \in \bar{B}) \leftrightarrow x \in \bar{A} \cap \bar{B}$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$x \in \overline{A \cap B} \leftrightarrow \neg(x \in A \wedge x \in B)$

$\leftrightarrow (\neg(x \in A) \vee \neg(x \in B))$

$\leftrightarrow (x \in \bar{A} \vee x \in \bar{B})$

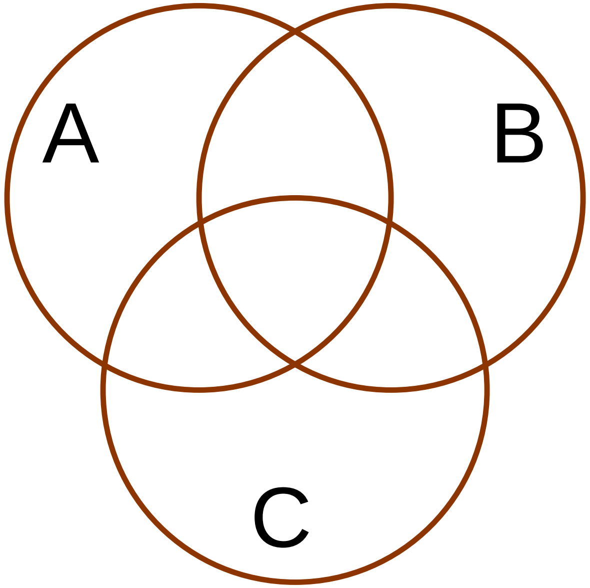
$\leftrightarrow x \in \bar{A} \cup \bar{B}$

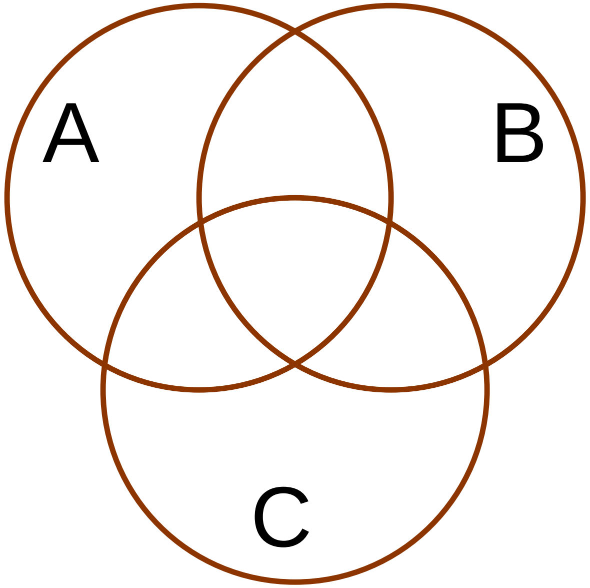
Proof technique:
To show C = D show
$x \in C \rightarrow x \in D$ and
$x \in D \rightarrow x \in C$

distributive laws

just like $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

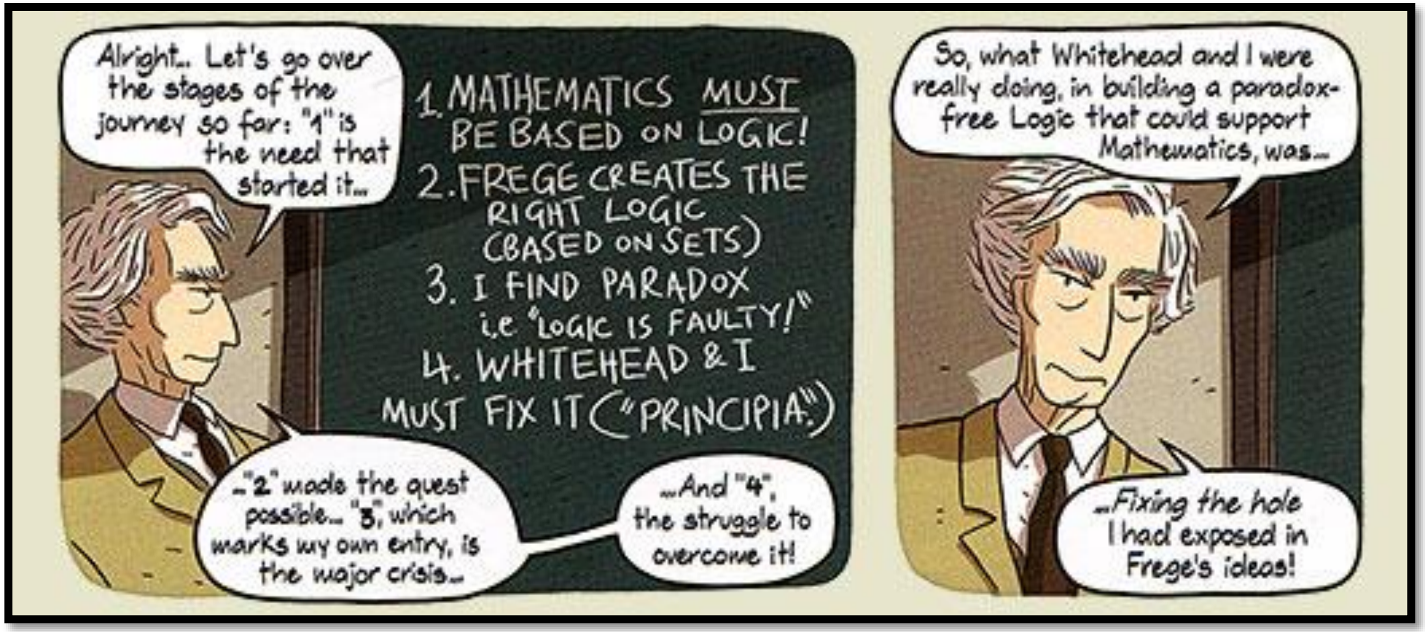
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





Russell's paradox

$$S = \{ x : x \notin x \}$$



representing sets using bits

- Suppose universe U is $\{1,2,\ldots,n\}$

- Can represent set $B \subseteq U$ as a vector of bits:

  $b_1 b_2 \cdots b_n$ where $b_i = 1$ when $i \in B$

  $b_i = 0$ when $i \notin B$

  – Called the *characteristic vector* of set B

- Given characteristic vectors for A and B
  – What is characteristic vector for $A \cup B$? $A \cap B$?

unix/linux file permissions

- **`ls –l`**

```
drwxr-xr-x ... Documents/
-rw-r--r-- ... file1
```

- Permissions maintained as bit vectors
  - Letter means bit is 1
  - “--” means bit is 0.

bitwise operations

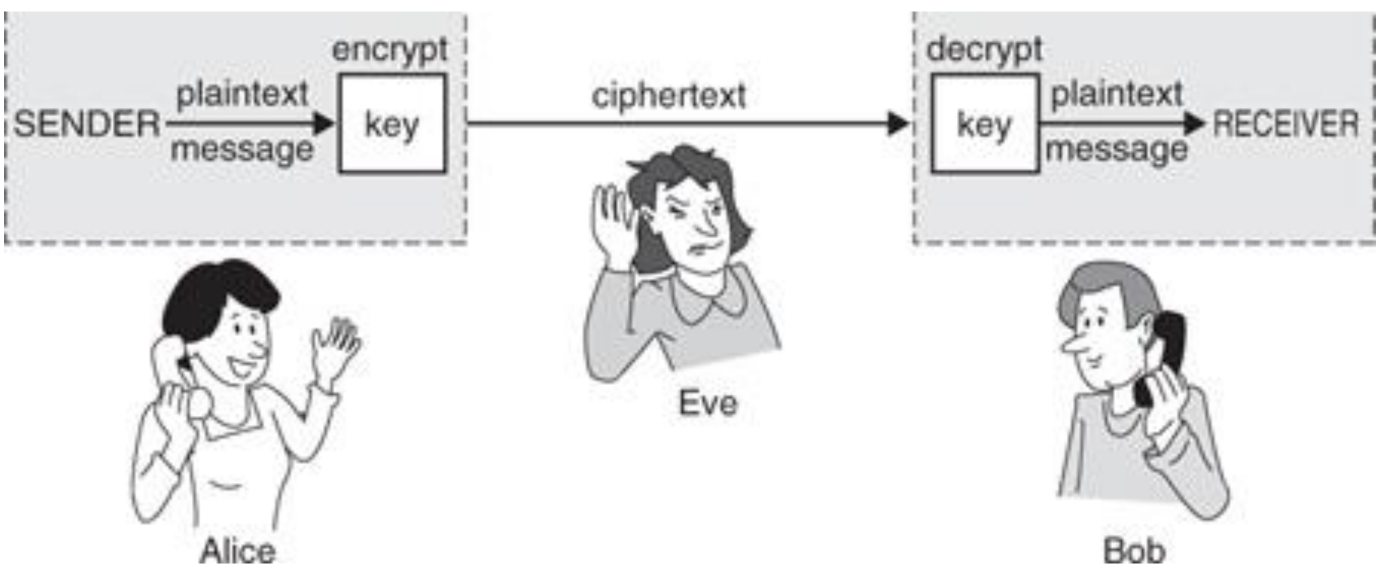
	01101101	Java:	`z=x|y`
$\vee$	00110111		
	01111111		
	00101010	Java:	`z=x&y`
$\wedge$	00001111		
	00001010		
	01101101	Java:	`z=x^y`
$\oplus$	00110111		
	01011010		

a useful identity

- If x and y are bits: $(x \oplus y) \oplus y = ?$ x
- What if x and y are bit-vectors? Same thing, bitwise

private key cryptography

- Alice wants to communicate message secretly to Bob so that eavesdropper Eve who hears their conversation cannot tell what Alice's message is.
- Alice and Bob can get together and privately share a secret key K ahead of time.



- Alice and Bob privately share random n-bit vector K
  - Eve does not know K
- Later, Alice has n-bit message m to send to Bob
  - Alice computes $C = m \oplus K$
  - Alice sends C to Bob
  - Bob computes $m = C \oplus K$ which is $(m \oplus K) \oplus K$
- Eve cannot figure out m from C unless she can guess K