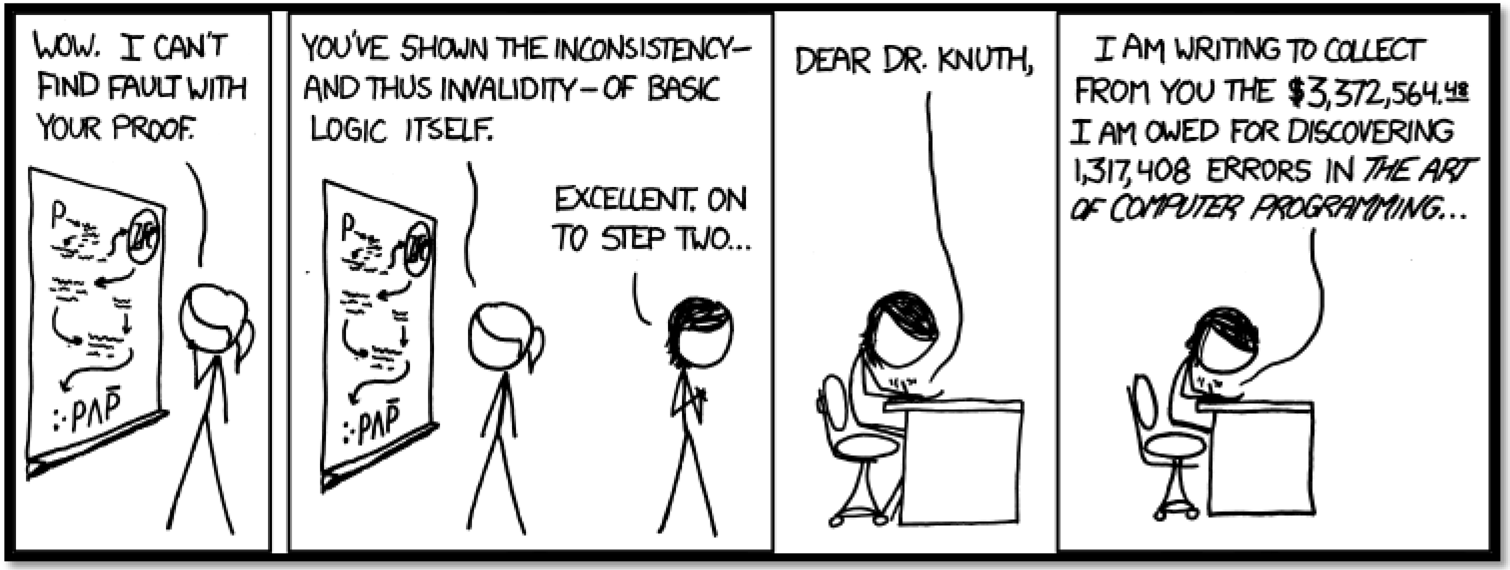
Fall 2015

Lecture 6: Predicate Logic, Logical Inference



quantifiers

$\forall x\ P(x)$

P(x) is true for every x in the domain

read as "for all x, P of x"

$\exists x\ P(x)$

There is an x in the domain for which P(x) is true

read as "there exists x, P of x"

negations of quantifiers

- not every positive integer is prime

  $\exists x \neg Prime(x) \equiv \neg \forall x\, Prime(x)$

  Domain: Pos. integers

- some positive integer is not prime

- prime numbers do not exist

- every positive integer is not prime

$\forall x$ PurpleFruit(x)

Domain:
Fruit

PurpleFruit(x)

Which one is equal to $\neg\forall x$ PurpleFruit(x)?

- $\exists x$ PurpleFruit(x)?
- $\exists x$ $\neg$PurpleFruit(x)?

de Morgan's laws for quantifiers

$$\neg \forall x \; P(x) \equiv \exists x \, \neg P(x)$$
$$\neg \exists x \; P(x) \equiv \forall x \, \neg P(x)$$

D = domain

$\forall x \; P(x) \equiv (P(x_1) \wedge P(x_2) \wedge \ldots)$

as $\{x_i\}$ ranges over D

$\exists x \; P(x) \equiv (P(x_1) \vee P(x_2) \vee \ldots)$

$\neg \forall x \, P(x) \equiv \neg (P(x_1) \wedge P(x_2) \wedge \ldots)$

$\equiv (\neg P(x_1) \vee \neg P(x_2) \vee \ldots)$

$\equiv \exists x \, \neg P(x)$

de Morgan's laws for quantifiers

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$
$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

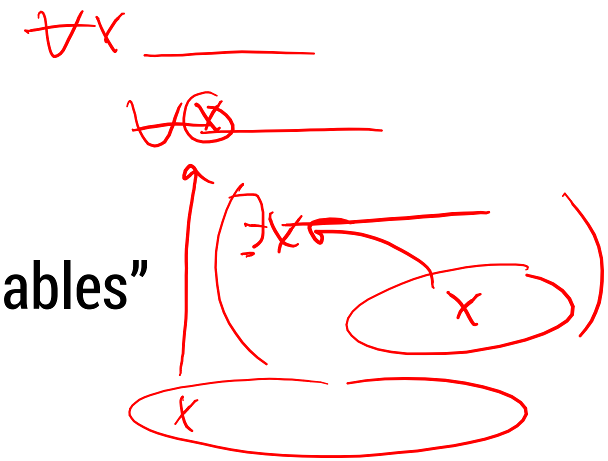
"There is no largest integer."

$$\neg\exists x\ \forall y\ (x \geq y)$$
$$\equiv \forall x\ \neg\forall y\ (x \geq y)$$
$$\equiv \forall x\ \exists y\ \neg(x \geq y)$$
$$\equiv \forall x\ \exists y\ (y > x)$$

"For every integer there is a larger integer."

scope of quantifiers

example: Notlargest(x) $\equiv \exists$ y Greater (y, x)
$\equiv \exists$ z Greater (z, x)



truth value:

doesn't depend on y or z "bound variables"

does depend on x "free variable"

quantifiers only act on free variables of the formula they quantify

$$\forall x\,(\exists y\,(P(x, y) \rightarrow \forall x\, Q(y, x)))$$

$$\equiv \forall x\,(\exists y\,(P(x,y) \rightarrow \forall z\, Q(y,z)))$$

$$\equiv \forall z\,(\exists y\, P(z,y) \rightarrow \forall x\, Q(y,x)))$$

example:

Domain = positive integers

$\text{IsMultiple}(x, y) =$ "x is a multiple of y"

$\forall x \left((x > 1 \land \neg(x = y)) \rightarrow \neg\text{IsMultiple}(y, x) \right)$

$\equiv \text{Prime}(y)$

$$\forall x \, \exists y \left((x < y) \land \text{Prime}(y) \right)$$

$$\forall x \, \exists y \left((x < y) \land \left(\forall x \left((x > 1 \land \neg(x = y)) \rightarrow \neg\text{IsMultiple}(y, x) \right) \right) \right)$$

example:

Domain = positive integers

$\text{IsMultiple}(x, y) =$ "x is a multiple of y"

$$\forall x\ \big((x > 1 \wedge \neg(x = y)) \to \neg\text{IsMultiple}(y, x)\big)$$

$$\equiv \text{Prime}(y)$$

$$\forall x\ \exists y\ \big((x < y) \wedge \text{Prime}(y) \wedge \text{Prime}(y + 2)\big)$$

$$\forall x\ \exists y \left(\begin{array}{c} (x < y) \wedge \Big(\forall x\ \big((x > 1 \wedge \neg(x = y)) \to \neg\text{IsMultiple}(y, x)\big)\Big) \\ \wedge \Big(\forall x\ \big((x > 1 \wedge \neg(x = \overset{y+2}{\cancel{y}})) \to \neg\text{IsMultiple}(\overset{y+2}{\cancel{y}}, x)\big)\Big) \end{array} \right)$$

scope of quantifiers

```
function f(x, y, z)
    x = y + z + a
    return x + y
```

example:

Domain = positive integers

$\text{IsMultiple}(x, y)$ = "x is a multiple of y"

$$\forall x\ \big((x > 1 \land \neg(x = y)) \rightarrow \neg\text{IsMultiple}(y, x)\big)$$

$$\equiv \text{Prime}(y)$$

$$\forall x\ \exists y\ \big((x < y) \land \text{Prime}(y) \land \text{Prime}(y + 2) \land (x < y^2)\big)$$

$$\forall x\ \exists y \left(\begin{array}{l} (x < y) \land \big(\forall x\ \big((x > 1 \land \neg(x = y)) \rightarrow \neg\text{IsMultiple}(y, x)\big)\big) \\ \land \big(\forall x\ \big((x > 1 \land \neg(x = \overset{y+2}{\cancel{y}})) \rightarrow \neg\text{IsMultiple}(\overset{y+2}{\cancel{y}}, x)\big)\big) \land (x < y^2) \end{array} \right)$$

scope of quantifiers

$\exists x\ (P(x) \land Q(x))$ vs. $(\exists x\ P(x)) \land (\exists x\ Q(x))$

Domain = {2000s pop stars}

$P(x)$ = x dated J. M.

$Q(x)$ = x is J. M.

$(\exists x\ P(x)) \land (\exists x\ Q(x)) \equiv T$

$\exists x\ (P(x) \land Q(x)) \equiv F$

nested quantifiers

- Bound variable names don't matter

$$\forall x \exists y\, P(x, y) \equiv \forall a \exists b\, P(a, b)$$

$$\not\equiv \forall x (\exists x\, P(x, x))$$

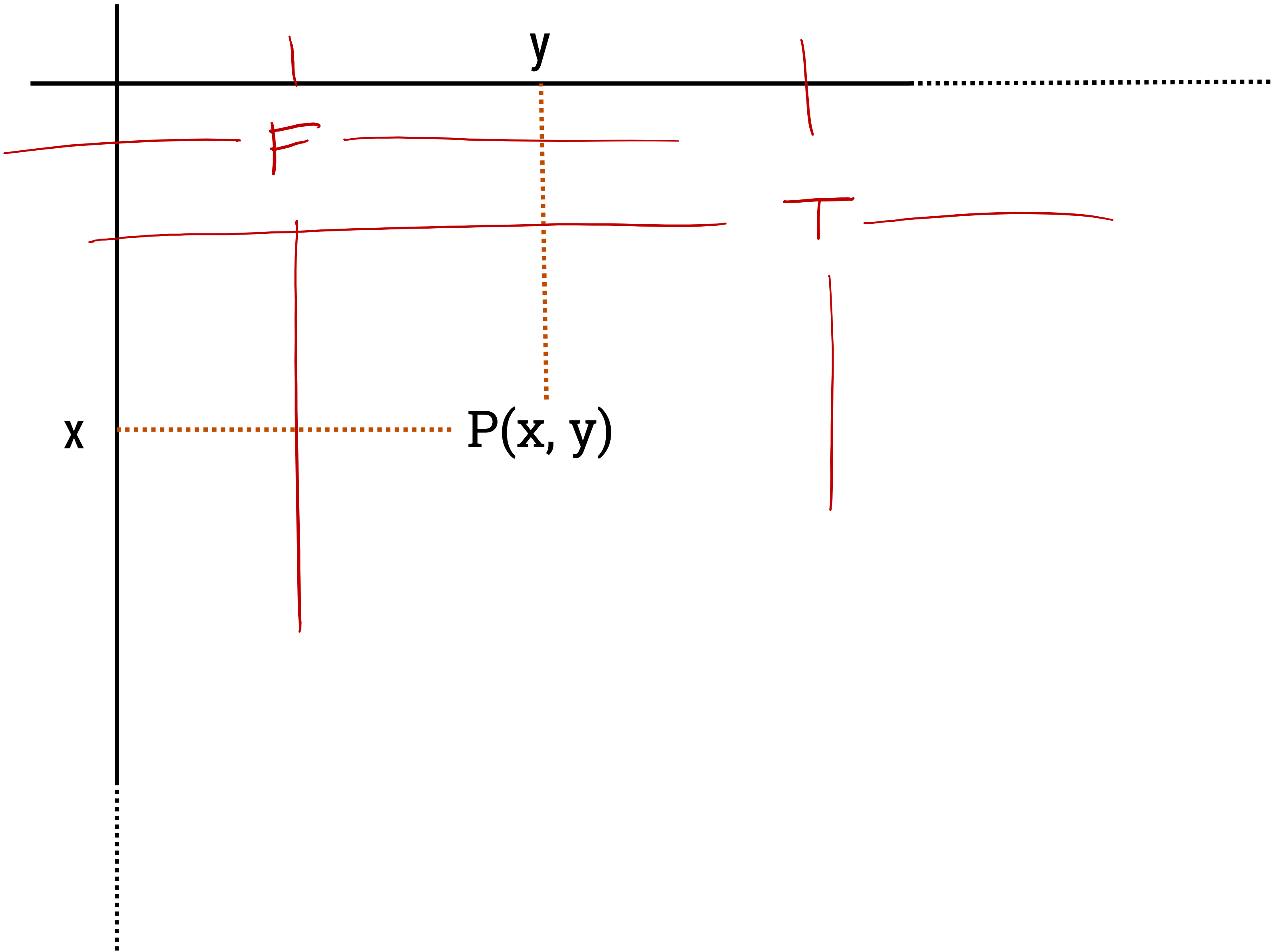
- Positions of quantifiers can sometimes change

$$\forall x\, (Q(x) \wedge \exists y\, P(x, y)) \equiv \forall x \exists y\, (Q(x) \wedge P(x, y))$$

- But: order is important...

$$\forall x \exists y \neq \exists y \forall x$$

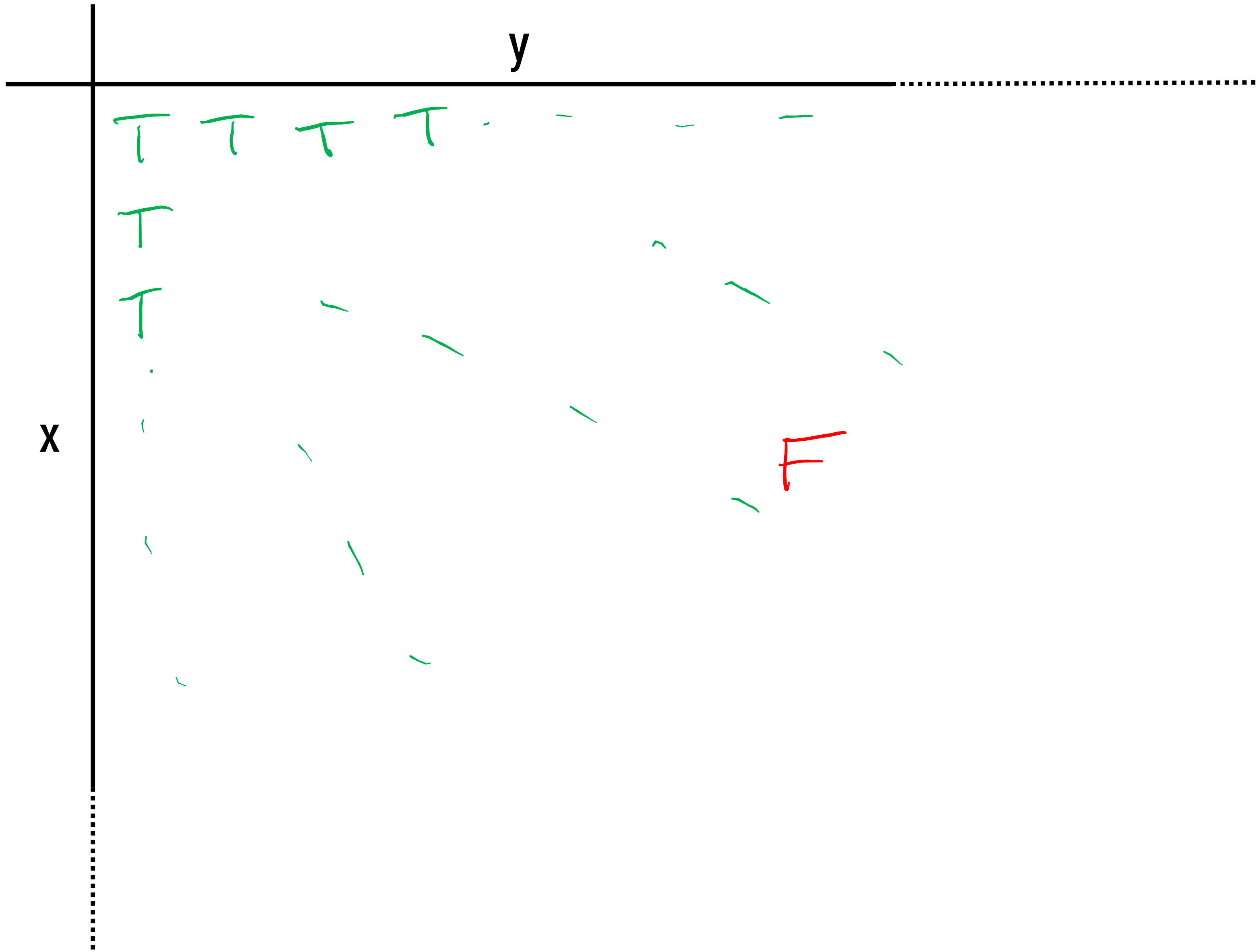
predicate with two variables



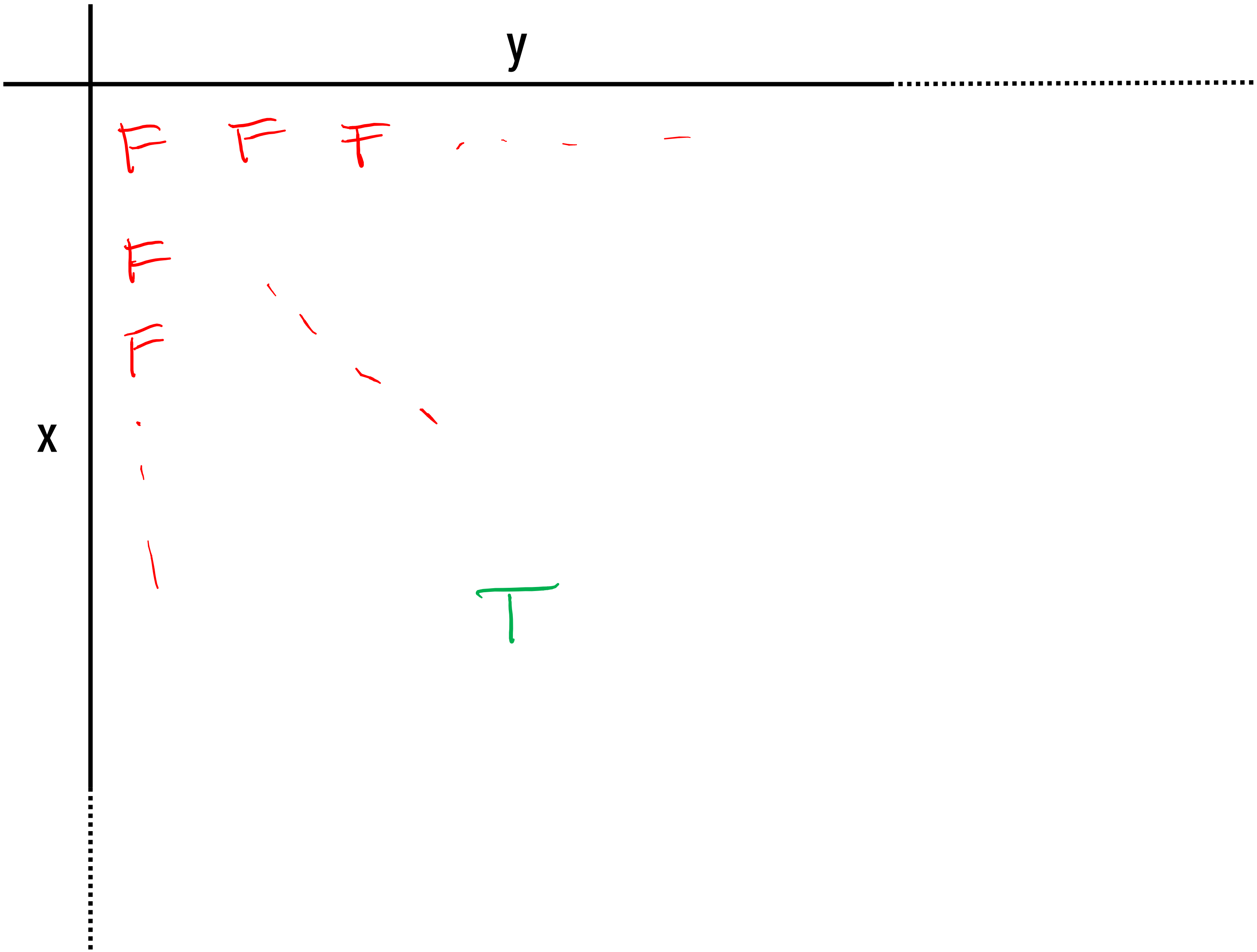
quantification with two variables

expression	**when true**	**when false**
$\forall x \forall y P(x, y)$		
$\exists x \exists y P(x, y)$		
$\forall x \exists y P(x, y)$		
$\exists x \forall y P(x, y)$		

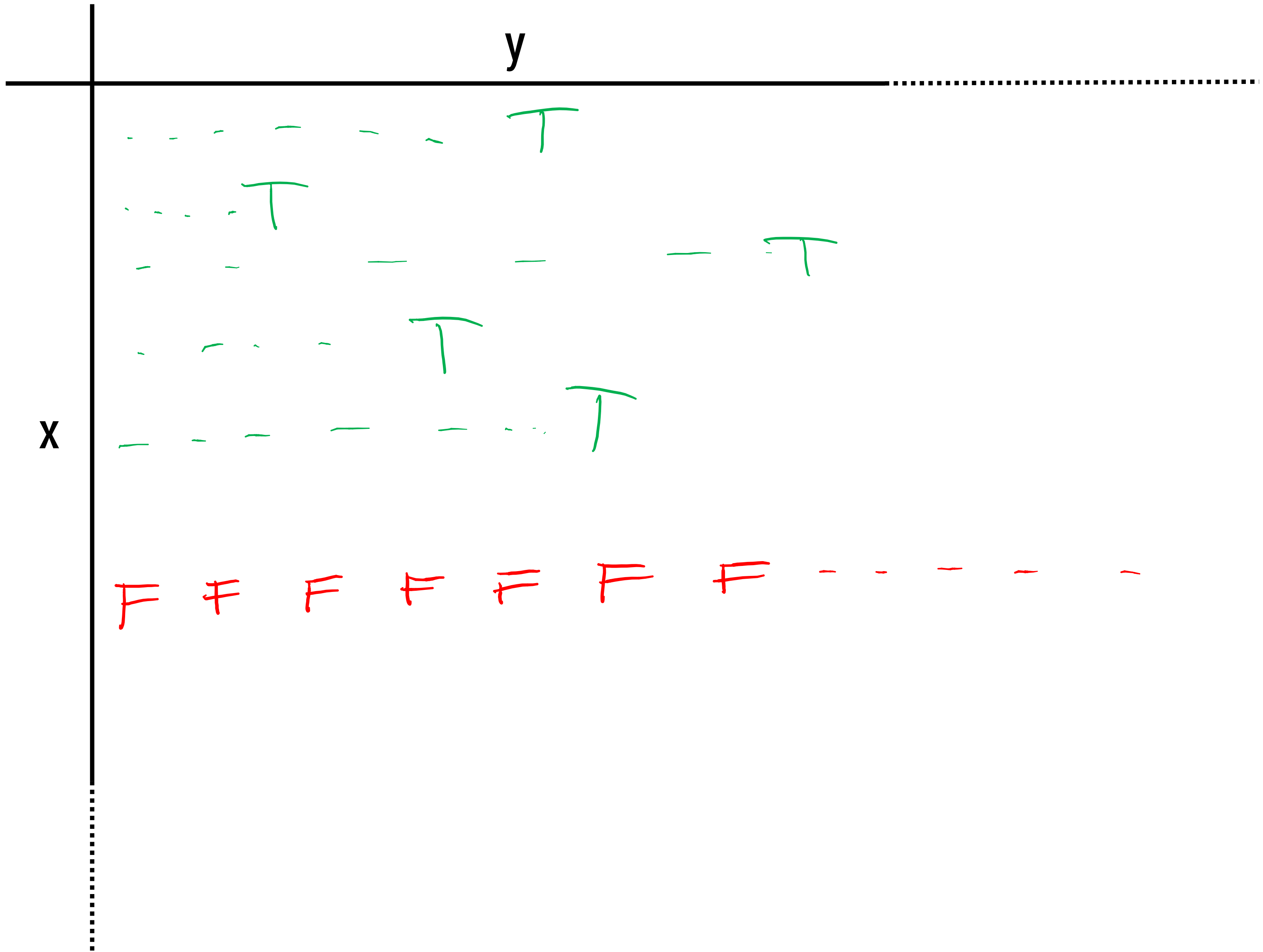
$\forall x\, \forall y\, P(x, y)$



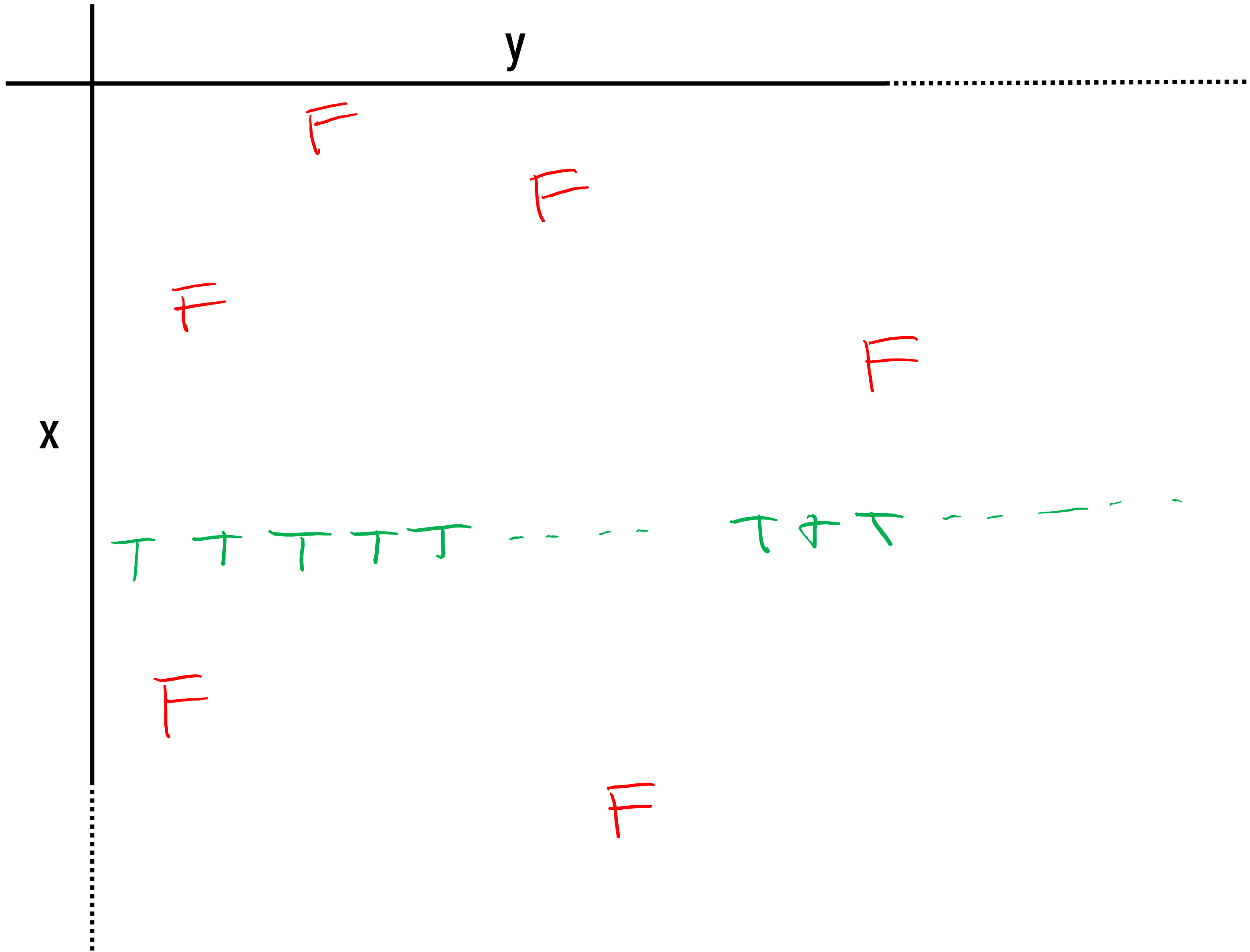
$\exists x\, \exists y\, P(x, y)$



$$\forall x \, \exists y \, P(x,y)$$



$\exists x \, \forall y \, P(x, y)$



quantification with two variables

expression	**when true**	**when false**
∀x ∀ y P(x, y)		
∃ x ∃ y P(x, y)		
∀ x ∃ y P(x, y)		
∃ x ∀ y P(x, y)		

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is only a small part of this

applications of logical inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

foundations of rational thought…

proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

an inference rule: *Modus Ponens*

- If p and p $\rightarrow$ q are both true then q must be true

- Write this rule as

$$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.

- Therefore, by modus ponens:
  - You have a 311 class today.

Show that r follows from p, $p \to q$, and $q \to r$

1.	p	given
2.	$p \to q$	given
3.	$q \to r$	given
4.	q	modus ponens from 1 and 2
5.	r	modus ponens from 3 and 4

proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	given
2.	$\neg q$	given
3.	$\neg q \rightarrow \neg p$	contrapositive of 1
4.	$\neg p$	modus ponens from 2 and 3

- Each inference rule is written as:

$$\frac{A, B}{\therefore C, D}$$

...which means that if both A and B are true then you can infer C and you can infer D.

  - For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies

- Sometimes rules don't need anything to start with. These rules are called axioms:
  - e.g. *Excluded Middle Axiom*

$$\frac{}{\therefore p \vee \neg p}$$

simple propositional inference rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it:

$$\frac{p \wedge q}{\therefore\ p,\ q} \qquad \frac{p,\ q}{\therefore\ p \wedge q}$$

$$\frac{p \vee q\ ,\ \neg p}{\therefore\ q} \qquad \frac{p}{\therefore\ p \vee q,\ q \vee p}$$

$$\frac{p,\ p \rightarrow q}{\therefore\ q} \qquad \frac{p \Rightarrow q}{\therefore\ p \rightarrow q}$$

Direct Proof Rule

Not like other rules

important: applications of inference rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise)

e.g. 1. $p \rightarrow q$ given

2. $(p \vee r) \rightarrow q$ intro $\vee$ from 1.

Does not follow! e.g. p=**F**, q=**F**, r=**T**

$p \rightarrow q \equiv \neg p \vee q$

3. $(p \rightarrow q) \rightarrow r$

4. $(\neg p \vee q) \rightarrow r$

logical equiv.

$$\frac{p}{\therefore\ p \vee r}$$

direct proof of an implication

- $p \Rightarrow q$ denotes a proof of q given p as an assumption
- The direct proof rule:

  If you have such a proof then you can conclude that $p \rightarrow q$ is true

Example:

proof subroutine

1. p **assumption**
2. $p \vee q$ intro for $\vee$ from 1
3. $p \rightarrow (p \vee q)$ direct proof rule