Fall 2015

Lecture 5: Canonical forms and predicate logic

Homework #1 Due Today at 11:59pm

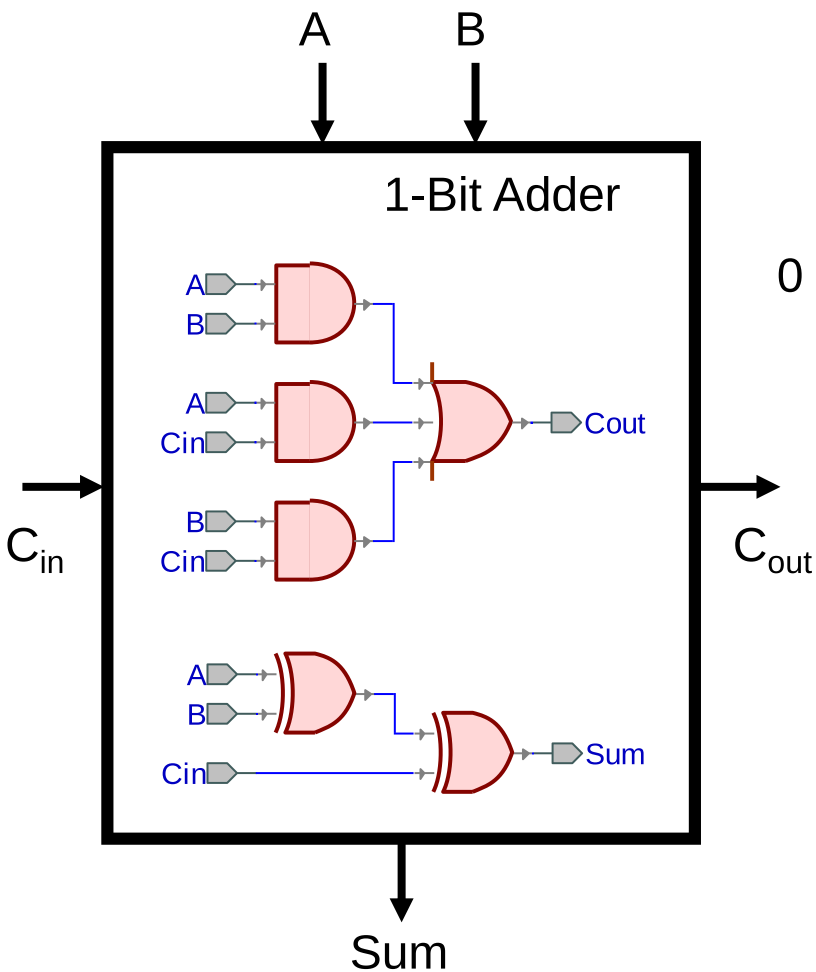
Your Gradescope login is your UW or CSE email address

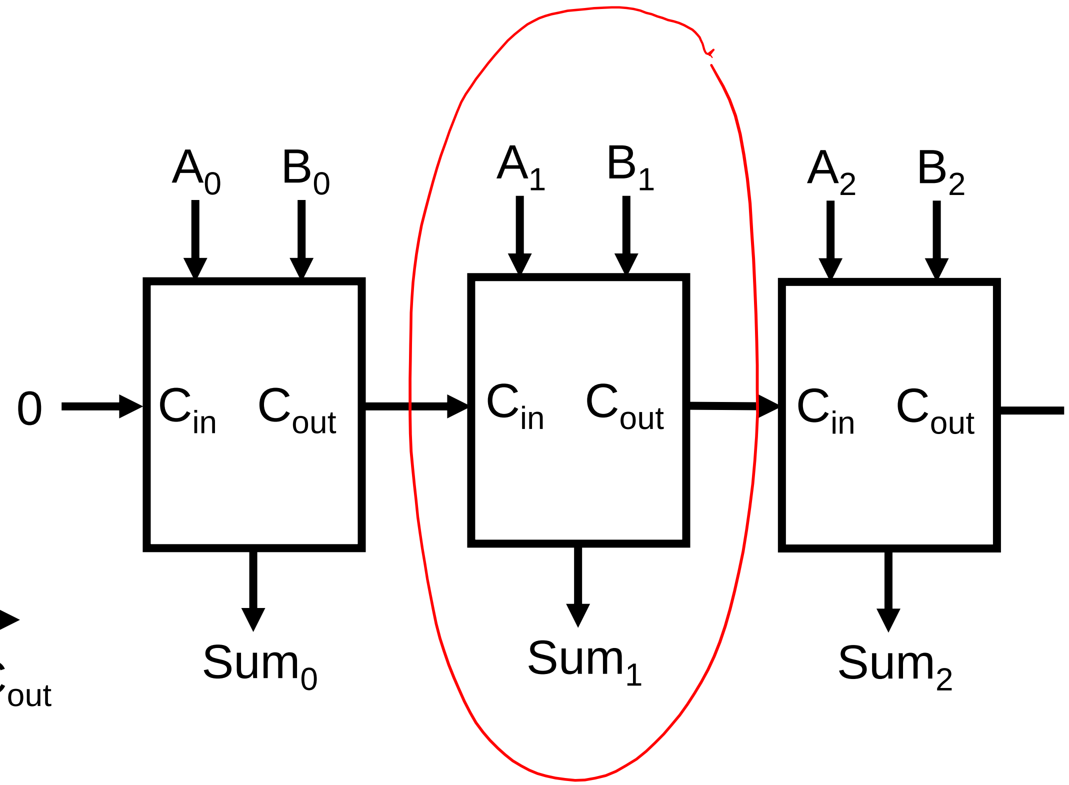
Homework #2 will be posted today and it is due next Friday

TA Office Hours

TA	**Office hours**	**Room**
Sam Castle	Wed, 12:00-1:00	CSE 021
Jiechen Chen	Tue, 4:00-5:00	CSE 218
Rebecca Leslie	Mon, 8:30-9:30	CSE 218
Evan McCarty	Tue, 11:30-12:30	CSE 220
Tim Oleskiw	Tue, 3:00-4:00	CSE 218
Spencer Peters	Tue, 1:00-2:00	CSE 218
Robert Weber	Wed, 3:30-4:30	CSE 678 (except Oct 21st at CSE 110)
Ian Zhu	Thu, 4:30-5:30	CSE 021

a 3-bit ripple-carry adder





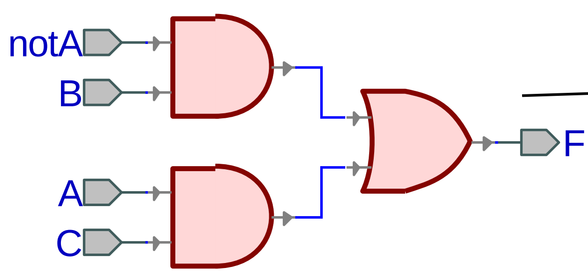
mapping truth tables to logic gates

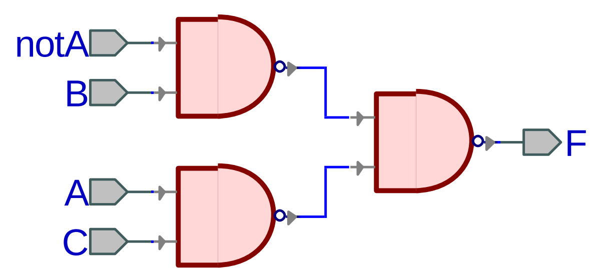
Given a truth table:

1. Write the Boolean expression
2. Minimize the Boolean expression
3. Draw as gates
4. Map to available gates

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$
\begin{aligned}
F &= A'BC' + A'BC + AB'C + ABC \\
&= A'B(C' + C) + AC(B' + B) \\
&= A'B + AC
\end{aligned}
$$





- Truth table is the unique signature of a Boolean function

- The same truth table can have many gate realizations
  - we've seen this already
  - depends on how good we are at Boolean simplification

- **Canonical forms**
  - standard forms for a Boolean expression
  - we all come up with the same expression

- also known as Disjunctive Normal Form (DNF)
- also known as minterm expansion

F = *001* *011* *101* *110* *111*

F = A'B'C + A'BC + AB'C + ABC' + ABC

A	B	C	F	F'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

F' = A'B'C' + A'BC' + AB'C'

sum-of-products canonical form

Product term (or minterm)

- ANDed product of literals – input combination for which output is true
- each variable appears exactly once, true or inverted (but not both)

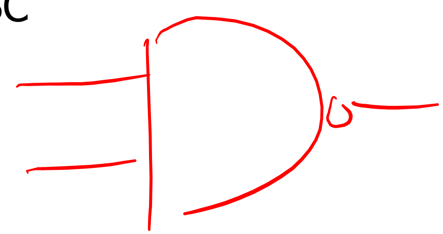
A	B	C	minterms
0	0	0	A′B′C′
0	0	1	A′B′C
0	1	0	A′BC′
0	1	1	A′BC
1	0	0	AB′C′
1	0	1	AB′C
1	1	0	ABC′
1	1	1	ABC

F in canonical form:

$$F(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= A'B'C + A'BC + AB'C + ABC + ABC' \\ &= (A'B' + A'B + AB' + AB)C + ABC' \\ &= ((A' + A)(B' + B))C + ABC' \\ &= C + ABC' \\ &= ABC' + C \\ &= AB + C \end{aligned}$$



- Also known as Conjunctive Normal Form (CNF)
- Also known as maxterm expansion

F = *000* *010* *100*

F = (A + B + C) (A + B′ + C) (A′ + B + C)

A	B	C	F	F′
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

$$F' = A'B'C' + A'BC' + AB'C'$$

$$F = (F')' = (A'B'C' + A'BC' + AB'C')'$$

$$= (A'B'C')'(A'BC')'(AB'C')'$$

$$= (A+B+C)(A+B'+C)(A'+B+C)$$

A B C

1 0 0

Complement of function in sum-of-products form:

- $F' = A'B'C' + A'BC' + AB'C'$

Complement again and apply de Morgan's and get the product-of-sums form:

- $(F')' = (A'B'C' + A'BC' + AB'C')'$
- $F = (A + B + C)\,(A + B' + C)\,(A' + B + C)$

p-o-s F ≡ de-Morgan (s-o-p F')

Sum term (or maxterm)

- ORed sum of literals – input combination for which output is false
- each variable appears exactly once

A	B	C	maxterms
0	0	0	$A+B+C$
0	0	1	$A+B+C'$
0	1	0	$A+B'+C$
0	1	1	$A+B'+C'$
1	0	0	$A'+B+C$
1	0	1	$A'+B+C'$
1	1	0	$A'+B'+C$
1	1	1	$A'+B'+C'$

F in canonical form:

$$F(A, B, C) = (A + B + C)\,(A + B' + C)\,(A' + B + C)$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= (A + B + C)\,(A + B' + C)\,(A' + B + C) \\ &= (A + B + C)\,(A + B' + C) \\ &\quad\ (A + B + C)\,(A' + B + C) \\ &= (A + C)\,(B + C) \end{aligned}$$

- Propositional Logic
  - If Pikachu doesn't wear pants, then he flies on Bieber's jet unless Taylor is feeling lonely.

- Predicate Logic
  - If x, y, and z are positive integers, then $x^3 + y^3 \neq z^3$. $P(x, y, z)$

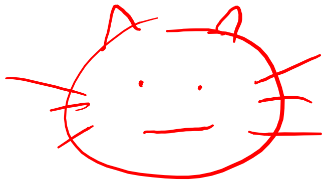
$P(x, y, z) =$ " $x^3 + y^3 \neq z^3$ "

$P(4, 5, 7) \equiv T$

Predicate or Propositional Function

- A function that returns a truth value, e.g.,

  "x is a cat"

  "x is prime"

  "student x has taken course y"

  "x > y"

  "x + y = z" or Sum(x, y, z)

  "5 < x"

Predicates will have variables or constants as arguments.

We must specify a "domain of discourse", which is the possible things we're talking about.

"x is a cat"

(e.g., mammals)

"x is prime"

(e.g., positive whole numbers)

"student x has taken course y"

(e.g., students and courses)

α

$$\forall x\ P(x)$$

P(x) is true for every x in the domain

read as "for all x, P of x"

$$\exists x\ P(x)$$

There is an x in the domain for which P(x) is true

read as "there exists x, P of x"

- $\exists x\ \mathrm{Even}(x)$ T
- $\forall x\ \mathrm{Odd}(x)$ F
- $\forall x\ (\mathrm{Even}(x) \vee \mathrm{Odd}(x))$ T
- $\exists x\ (\mathrm{Even}(x) \wedge \mathrm{Odd}(x))$ F
- $\forall x\ \mathrm{Greater}(x+1, x)$ T
- $\exists x\ (\mathrm{Even}(x) \wedge \mathrm{Prime}(x))$ T

Domain:
Positive Integers

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
(or "x>y")
Equal(x,y)
(or "x=y")
Sum(x,y,z)
(or "z=x+y")

$(x=2)$

$\boxed{\exists x \quad P(x)}$

s.t. = such that

statements with quantifiers

Domain:

Positive Integers

- $\forall x\ \exists y$ Greater (y, x) T $(y = x+1)$
- $\forall x\ \exists y$ Greater (x, y) F $x = 1$
- $\forall x\ \exists y$ (Greater$(y, x) \wedge$ Prime(y)) T
- $\forall x$ (Prime$(x) \rightarrow$ (Equal$(x, 2) \vee$ Odd(x)))) T "$\forall x \in \mathbb{N}$"
- $\exists x\ \exists y$ (Sum$(x, 2, y) \wedge$ Prime$(x) \wedge$ Prime(y)) T $x = 3$ $y = 5$

Even(x)

Odd(x)

Prime(x)

Greater(x,y)

(or "x>y")

Equal(x,y)

(or "x=y")

Sum(x,y,z)

(or "z=x+y")

	Prev	Now
• $\forall x\ \exists y$ Greater (y, x)	T	T
• $\forall x\ \exists y$ Greater (x, y)	F	T

$y = x - 1$

Domain:

All integers

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
(or "$x>y$")
Equal(x,y)
(or "$x=y$")
Sum(x,y,z)
(or "$z=x+y$")

Domain of quantifiers is important!

$F \rightarrow T \equiv T$

Cat(x)
Red(x)
LikesTofu(x)

- "Red cats like tofu"

$\cancel{\forall x (Cat(x) \wedge Red(x) \wedge LikesTofu(x))}$

$\forall x ((Red(x) \wedge Cat(x)) \rightarrow LikesTofu(x))$

- "Some red cats don't like tofu"

$\exists x (Cat(x) \wedge Red(x) \wedge \neg LikesTofu(x))$

wrong: $\exists x ((Cat(x) \wedge Red(x)) \rightarrow \neg LikesTofu(x))$

All cats like tofu and there is a red and a green cat.

$\exists x\ P(x)$

- not every positive integer is prime

  $\exists x\ \neg \text{Prime}(x)$
  $\equiv \neg(\forall x\ \text{Prime}(x))$

Domain
pos. integers

Prime(x)

- some positive integer is not prime

- prime numbers do not exist

- every positive integer is not prime

$\forall x$ PurpleFruit(x)

Domain:

Fruit

PurpleFruit(x)

Which one is equal to $\neg\forall x$ PurpleFruit(x)?

- $\exists x$ PurpleFruit(x)?

- $\exists x$ $\neg$PurpleFruit(x)?

$$\neg \forall x \ P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x \ P(x) \equiv \forall x \neg P(x)$$

$$\neg \forall x \, P(x) \equiv \exists x \, \neg P(x)$$

$$\forall x \ P(x) = (P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \cdots)$$

$\{x_i\}$ ranges over the domain

$$\neg \forall x \, P(x) \equiv \neg (P(x_1) \wedge \cdots\cdots)$$
$$\equiv \neg P(x_1) \vee \neg P(x_2) \vee \cdots\cdots$$

$$\boxed{\begin{array}{l} \neg\forall x\ P(x) \equiv \exists x\ \neg P(x) \\ \neg\exists x\ P(x) \equiv \forall x\ \neg P(x) \end{array}}$$

"There is no largest integer."

$$\begin{aligned} & \neg\exists x\ \forall y\ (x \geq y) \\ \equiv\ & \forall x\ \neg\forall y\ (x \geq y) \\ \equiv\ & \forall x\ \exists y\ \neg(x \geq y) \\ \equiv\ & \forall x\ \exists y\ (y > x) \end{aligned}$$

"For every integer there is a larger integer."