Homework #1 Due Friday at 11:59pm

Please try out Gradescope before then!

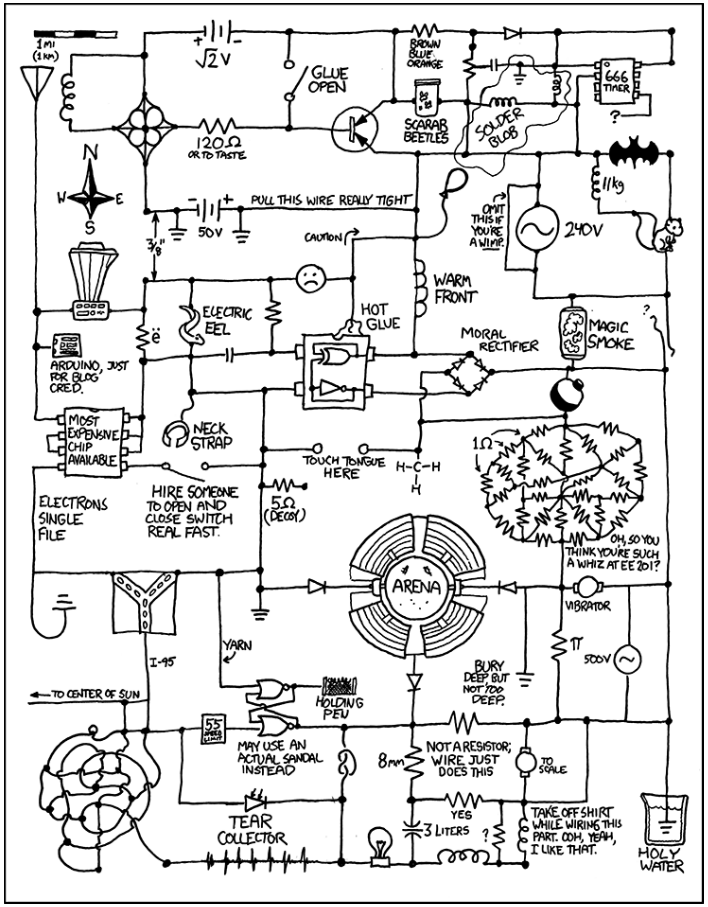
(You can submit multiple times, so do a test run on the first homework.)

Sections start this week:

Section	Day/Time	Room
AA Sam	Th, 830-920	MGH 242
AB Rebecca	Th, 930-1020	MGH 234
AC Robert	Th, 1030-1120	JHN 075
BA Jiechen	Th, 1230-120	MGH 228
BB Tim	Th, 130-220	MGH 242
BC Evan	Th, 230-320	MEB 242

Spring 2015

Lecture 4: Boolean Algebra and Circuits



a combinatorial logic example

Sessions of class:

We would like to compute the number of lectures or quiz sections remaining *at the start* of a given day of the week.

- Inputs: Day of the Week, Lecture/Section flag
- Output: Number of sessions left

Examples: Input: (Wednesday, Lecture) Output: 2
Input: (Monday, Section) Output: 1

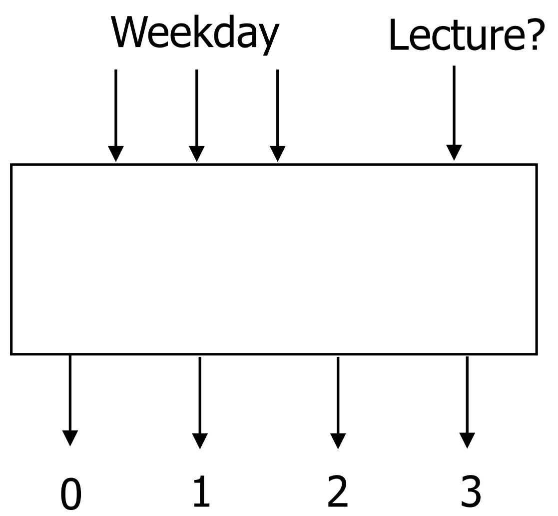
implementation in software

```
public int classesLeft (weekday, lecture_flag) {
    switch (day) {
        case SUNDAY:
        case MONDAY:
            return lecture_flag ? 3 : 1;
        case TUESDAY:
        case WEDNESDAY:
            return lecture_flag ? 2 : 1;
        case THURSDAY:
            return lecture_flag ? 1 : 1;
        case FRIDAY:
            return lecture_flag ? 1 : 0;
        case SATURDAY:
            return lecture_flag ? 0 : 0;
   }
}
```

implementation with combinational logic

Encoding:

- How many bits for each input/output?
- Binary number for weekday
- One bit for each possible output



defining our inputs

```
public int classesLeft (weekday, lecture_flag) {
    switch (day) {
        case SUNDAY:
        case MONDAY:
            return lecture_flag ? 3 : 1;
        case TUESDAY:
        case WEDNESDAY:
             return lecture_flag ? 2 : 1;
        case THURSDAY:
             return lecture_flag ? 1 : 1;
        case FRIDAY:
             return lecture_flag ? 1 : 0;
        case SATURDAY:
             return lecture_flag ? 0 : 0;
    }
}
```

Weekday	Number	Binary
Sunday	0	$(000)_2$
Monday	1	$(001)_2$
Tuesday	2	$(010)_2$
Wednesday	3	$(011)_2$
Thursday	4	$(100)_2$
Friday	5	$(101)_2$
Saturday	6	$(110)_2$

converting to a truth table

Weekday	Number	Binary
Sunday	0	$(000)_2$
Monday	1	$(001)_2$
Tuesday	2	$(010)_2$
Wednesday	3	$(011)_2$
Thursday	4	$(100)_2$
Friday	5	$(101)_2$
Saturday	6	$(110)_2$

Weekday	Lecture?	c0	c1	c2	c3
000	0	0	1	0	0
000	1	0	0	0	1
001	0	0	1	0	0
001	1	0	0	0	1
010	0	0	1	0	0
010	1	0	0	1	0
011	0	0	1	0	0
011	1	0	0	1	0
100	-	0	1	0	0
101	0	1	0	0	0
101	1	0	1	0	0
110	-	1	0	0	0
111	-	-	-	-	-

truth table ⇒ logic (part one)

c3 = (DAY == SUN and LEC) or (DAY == MON and LEC)

c3 = (d2 == 0 && d1 == 0 && d0 == 0 && L == 1) ||
(d2 == 0 && d1 == 0 && d0 == 1 && L == 1)

c3 = d2'•d1'•d0'•L + d2'•d1'•d0•L

DAY	d2d1d0	L	c0	c1	c2	c3
SunS	000	0	0	1	0	0
SunL	000	1	0	0	0	1
MonS	001	0	0	1	0	0
MonL	001	1	0	0	0	1
TueS	010	0	0	1	0	0
TueL	010	1	0	0	1	0
WedS	011	0	0	1	0	0
WedL	011	1	0	0	1	0
Thu	100	-	0	1	0	0
FriS	101	0	1	0	0	0
FriL	101	1	0	1	0	0
Sat	110	-	1	0	0	0
-	111	-	-	-	-	-

truth table ⇒ logic (part two)

c3 = d2’•d1’•d0’•L + d2’•d1’•d0•L

c2 = (DAY == TUE and LEC) or
(DAY == WED and LEC)

c2 = d2’•d1•d0’•L + d2’•d1•d0•L

DAY	d2d1d0	L	c0	c1	c2	c3
SunS	000	0	0	1	0	0
SunL	000	1	0	0	0	1
MonS	001	0	0	1	0	0
MonL	001	1	0	0	0	1
TueS	010	0	0	1	0	0
TueL	010	1	0	0	1	0
WedS	011	0	0	1	0	0
WedL	011	1	0	0	1	0
Thu	100	-	0	1	0	0
FriS	101	0	1	0	0	0
FriL	101	1	0	1	0	0
Sat	110	-	1	0	0	0
-	111	-	-	-	-	-

truth table ⇒ logic (part three)

$c3 = d2'\cdot d1'\cdot d0'\cdot L + d2'\cdot d1'\cdot d0\cdot L$

$c2 = d2'\cdot d1\cdot d0'\cdot L + d2'\cdot d1\cdot d0\cdot L$

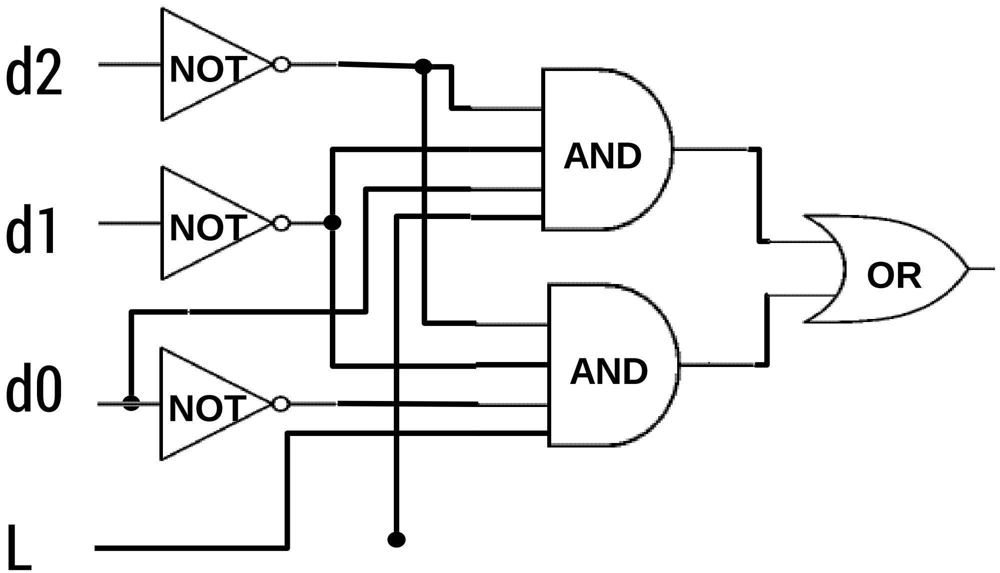
$c1 =$

[you do this one]

$c0 = d2\cdot d1'\cdot d0\cdot L' + d2\cdot d1\cdot d0'$

DAY	d2d1d0	L	c0	c1	c2	c3
SunS	000	0	0	1	0	0
SunL	000	1	0	0	0	1
MonS	001	0	0	1	0	0
MonL	001	1	0	0	0	1
TueS	010	0	0	1	0	0
TueL	010	1	0	0	1	0
WedS	011	0	0	1	0	0
WedL	011	1	0	0	1	0
Thu	100	-	0	1	0	0
FriS	101	0	1	0	0	0
FriL	101	1	0	1	0	0
Sat	110	-	1	0	0	0
-	111	-	-	-	-	-

$c3 = d2'•d1'•d0'•L + d2'•d1'•d0•L$

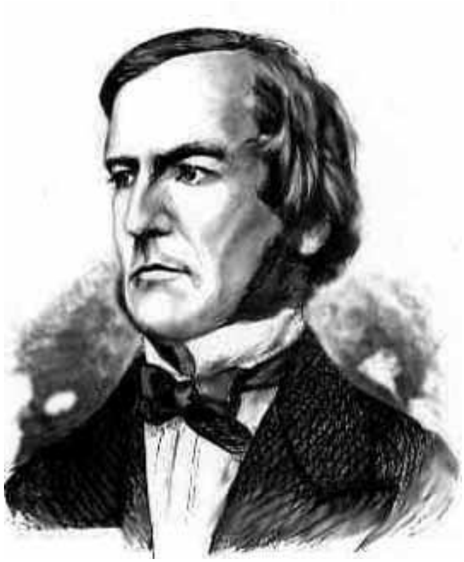


(multiple input AND gates)

DAY	d2d1d0	L	c0	c1	c2	c3
SunS	000	0	0	1	0	0
SunL	000	1	0	0	0	1
MonS	001	0	0	1	0	0
MonL	001	1	0	0	0	1
TueS	010	0	0	1	0	0
TueL	010	1	0	0	1	0
WedS	011	0	0	1	0	0
WedL	011	1	0	0	1	0
Thu	100	-	0	1	0	0
FriS	101	0	1	0	0	0
FriL	101	1	0	1	0	0
Sat	110	-	1	0	0	0
-	111	-	-	-	-	-

Boolean algebra

- Boolean algebra to circuit design



- Boolean algebra
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { ’ }
  - such that the following axioms hold:

1. The set B contains at least two elements: 0, 1

For any a, b, c in B:

2. closure:	$a + b$ is in B	$a \cdot b$ is in B
3. commutativity:	$a + b = b + a$	$a \cdot b = b \cdot a$
4. associativity:	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
5. identity:	$a + 0 = a$	$a \cdot 1 = a$
6. distributivity:	$a + (b \cdot c) = (a + b) \cdot (a + c)$	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
7. complementarity:	$a + a' = 1$	$a \cdot a' = 0$

axioms and theorems of Boolean algebra

identity:

1. $X + 0 = X$ — 1D. $X \cdot 1 = X$

null:

2. $X + 1 = 1$ — 2D. $X \cdot 0 = 0$

idempotency:

3. $X + X = X$ — 3D. $X \cdot X = X$

involution:

4. $(X')' = X$

complementarity:

5. $X + X' = 1$ — 5D. $X \cdot X' = 0$

commutativity:

6. $X + Y = Y + X$ — 6D. $X \cdot Y = Y \cdot X$

associativity:

7. $(X + Y) + Z = X + (Y + Z)$ — 7D. $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$

distributivity:

8. $X \cdot (Y + Z) = (X \cdot Y) + (X \cdot Z)$ — 8D. $X + (Y \cdot Z) = (X + Y) \cdot (X + Z)$

axioms and theorems of Boolean algebra

uniting:

9. $X \cdot Y + X \cdot Y' = X$

9D. $(X + Y) \cdot (X + Y') = X$

absorption:

10. $X + X \cdot Y = X$

10D. $X \cdot (X + Y) = X$

11. $(X + Y') \cdot Y = X \cdot Y$

11D. $(X \cdot Y') + Y = X + Y$

factoring:

12. $(X + Y) \cdot (X' + Z) = X \cdot Z + X' \cdot Y$

12D. $X \cdot Y + X' \cdot Z = (X + Z) \cdot (X' + Y)$

consensus:

13. $(X \cdot Y) + (Y \cdot Z) + (X' \cdot Z) = X \cdot Y + X' \cdot Z$

13D. $(X + Y) \cdot (Y + Z) \cdot (X' + Z) = (X + Y) \cdot (X' + Z)$

de Morgan's:

14. $(X + Y + \ldots)' = X' \cdot Y' \cdot \ldots$

14D. $(X \cdot Y \cdot \ldots)' = X' + Y' + \ldots$

Using the laws of Boolean Algebra:

prove the theorem: $X \bullet Y + X \bullet Y' = X$

distributivity (8)	$X \bullet Y + X \bullet Y' = X \bullet (Y + Y')$
complementarity (5)	$= X \bullet (1)$
identity (1D)	$= X$

prove the theorem: $X + X \bullet Y = X$

identity (1D)	$X + X \bullet Y = X \bullet 1 + X \bullet Y$
distributivity (8)	$= X \bullet (1 + Y)$
uniting (2)	$= X \bullet (1)$
identity (1D)	$= X$

proving theorems (truth table)

Using complete truth table:

For example, de Morgan's Law:

$(X + Y)' = X' \bullet Y'$
NOR is equivalent to AND
with inputs complemented

X	Y	X′	Y′	(X + Y)′	X′ • Y′
0	0	1	1		
0	1	1	0		
1	0	0	1		
1	1	0	0		

$(X \bullet Y)' = X' + Y'$
NAND is equivalent to OR
with inputs complemented

X	Y	X′	Y′	(X • Y)′	X′ + Y′
0	0	1	1		
0	1	1	0		
1	0	0	1		
1	1	0	0		

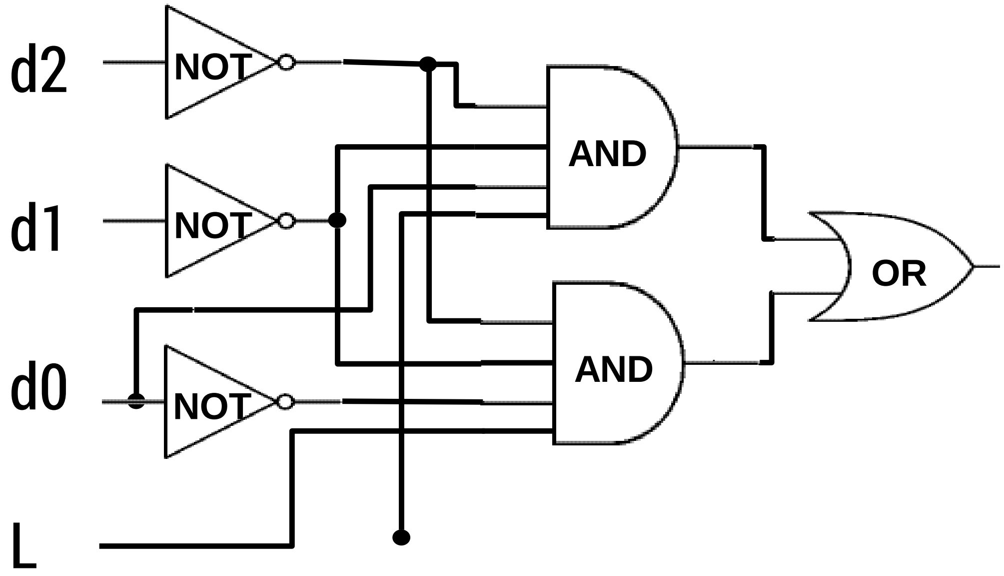
simplifying using Boolean algebra

c3 = d2'•d1'•d0'•L + d2'•d1'•d0•L

= d2'•d1'•(d0' + d0)•L

= d2'•d1'•(1)•L

= d2'•d1'•L



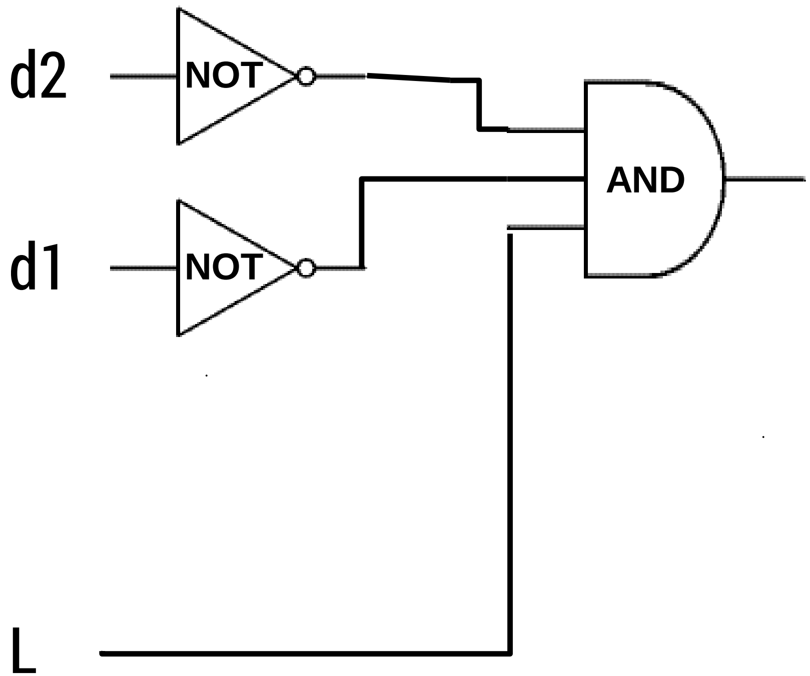
simplifying using Boolean algebra

c3 = d2'•d1'•d0'•L + d2'•d1'•d0•L

= d2'•d1'•(d0' + d0)•L

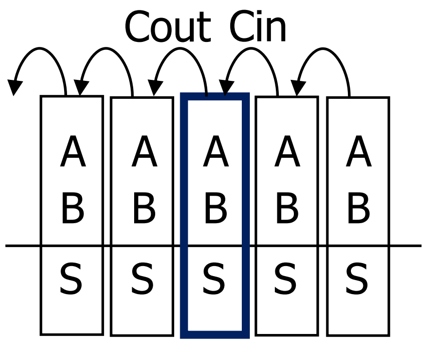
= d2'•d1'•(1)•L

= d2'•d1'•L



1-bit binary adder

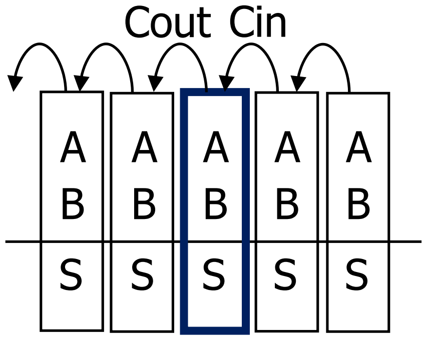
- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	Cin	Cout	S
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

1-bit binary adder

- Inputs: A, B, Carry-in
- Outputs: Sum, Carry-out



A	B	Cin	Cout	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$S = A' B' Cin + A' B Cin' + A B' Cin' + A B Cin$$

$$Cout = A' B Cin + A B' Cin + A B Cin' + A B Cin$$

apply theorems to simplify expressions

The theorems of Boolean algebra can simplify expressions

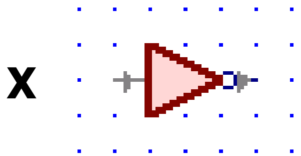
– e.g., full adder's carry-out function

$$
\begin{aligned}
\text{Cout} &= A'\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= A'\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} + A\,B\,\text{Cin} \\
&= A'\,B\,\text{Cin} + A\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= (A' + A)\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= (1)\,B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} + A\,B\,\text{Cin} \\
&= B\,\text{Cin} + A\,B'\,\text{Cin} + A\,B\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= B\,\text{Cin} + A\,(B' + B)\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= B\,\text{Cin} + A\,(1)\,\text{Cin} + A\,B\,\text{Cin}' + A\,B\,\text{Cin} \\
&= B\,\text{Cin} + A\,\text{Cin} + A\,B\,(\text{Cin}' + \text{Cin}) \\
&= B\,\text{Cin} + A\,\text{Cin} + A\,B\,(1) \\
&= B\,\text{Cin} + A\,\text{Cin} + A\,B
\end{aligned}
$$

adding extra terms creates new factoring opportunities

NOT

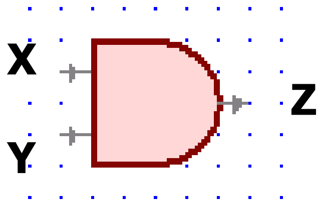
X' $\quad \bar{X}$ $\quad \neg X$



X	Y
0	1
1	0

AND

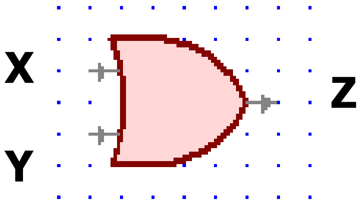
$X \cdot Y$ $\quad XY$ $\quad X \wedge Y$



X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

OR

$X + Y$ $\quad X \vee Y$

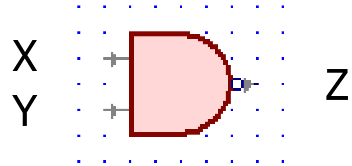


X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

more gates

NAND

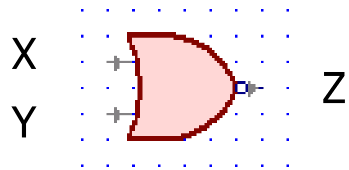
$\neg(X \wedge Y) \quad (XY)'$



X	Y	Z
0	0	1
0	1	1
1	0	1
1	1	0

NOR

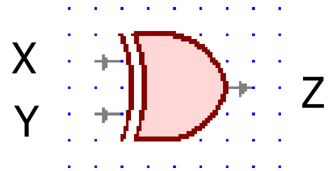
$\neg(X \vee Y) \quad (X + Y)'$



X	Y	Z
0	0	1
0	1	0
1	0	0
1	1	0

XOR

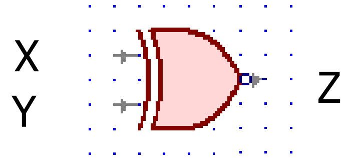
$X \oplus Y$



X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	0

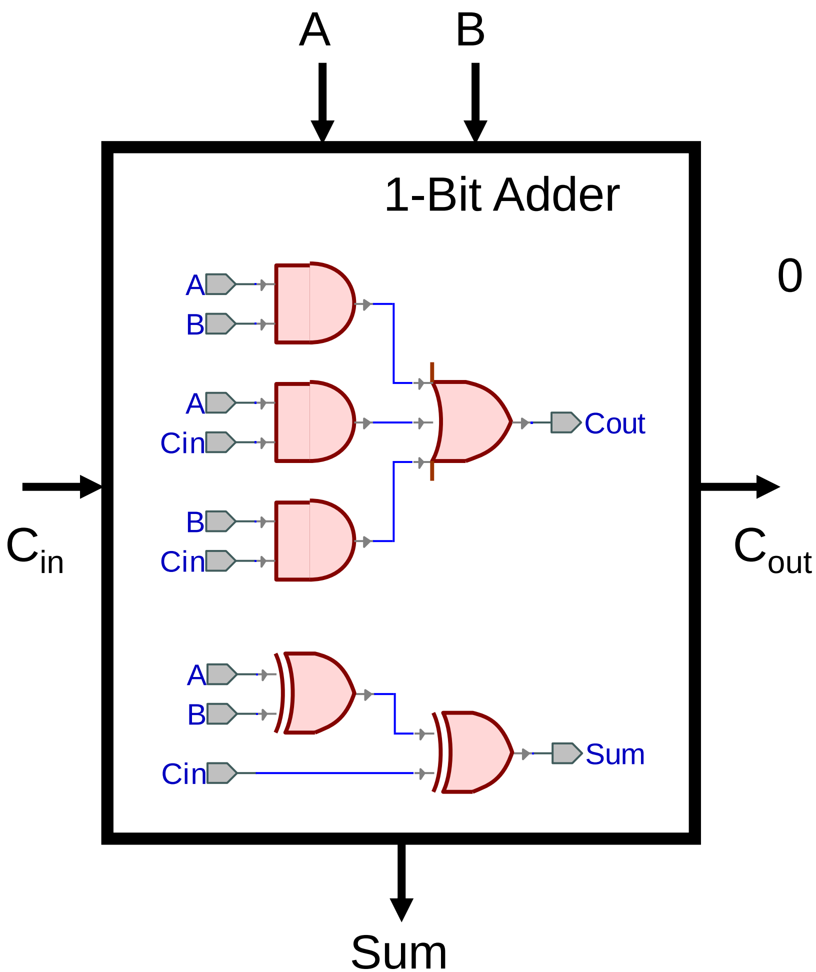
XNOR

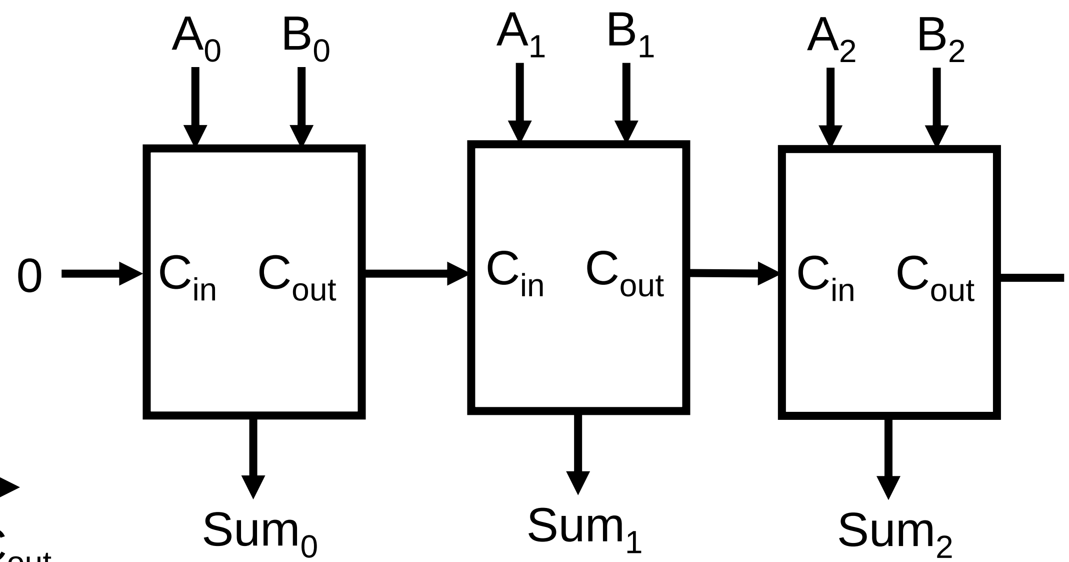
$X \leftrightarrow Y$



X	Y	Z
0	0	1
0	1	0
1	0	0
1	1	1

a 2-bit ripple-carry adder





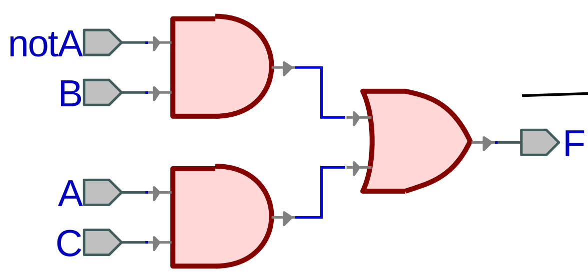
mapping truth tables to logic gates

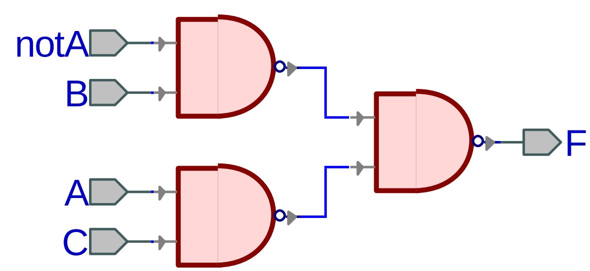
Given a truth table:

1. Write the Boolean expression
2. Minimize the Boolean expression
3. Draw as gates
4. Map to available gates

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$F = A'BC'+A'BC+AB'C+ABC$$
$$= A'B(C'+C)+AC(B'+B)$$
$$= A'B+AC$$





- Truth table is the unique signature of a Boolean function

- The same truth table can have many gate realizations
  - we've seen this already
  - depends on how good we are at Boolean simplification

- **Canonical forms**
  - standard forms for a Boolean expression
  - we all come up with the same expression

sum-of-products canonical form

- also known as Disjunctive Normal Form (DNF)
- also known as minterm expansion

F = *001* *011* *101* *110* *111*

$F = A'B'C + A'BC + AB'C + ABC' + ABC$

A	B	C	F	F′
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

sum-of-products canonical form

Product term (or minterm)

- ANDed product of literals – input combination for which output is true
- each variable appears exactly once, true or inverted (but not both)

A	B	C	minterms
0	0	0	A′B′C′
0	0	1	A′B′C
0	1	0	A′BC′
0	1	1	A′BC
1	0	0	AB′C′
1	0	1	AB′C
1	1	0	ABC′
1	1	1	ABC

F in canonical form:

$$F(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= A'B'C + A'BC + AB'C + ABC + ABC' \\ &= (A'B' + A'B + AB' + AB)C + ABC' \\ &= ((A' + A)(B' + B))C + ABC' \\ &= C + ABC' \\ &= ABC' + C \\ &= AB + C \end{aligned}$$

product-of-sums canonical form

- Also known as Conjunctive Normal Form (CNF)
- Also known as maxterm expansion

F = *000* *010* *100*

$F = (A + B + C)\,(A + B' + C)\,(A' + B + C)$

A	B	C	F	F′
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

Complement of function in sum-of-products form:

– $F' = A'B'C' + A'BC' + AB'C'$

Complement again and apply de Morgan's and get the product-of-sums form:

– $(F')' = (A'B'C' + A'BC' + AB'C')'$

– $F = (A + B + C)\ (A + B' + C)\ (A' + B + C)$

product-of-sums canonical form

Sum term (or maxterm)

- ORed sum of literals – input combination for which output is false
- each variable appears exactly once, true or inverted (but not both)

A	B	C	maxterms
0	0	0	A+B+C
0	0	1	A+B+C′
0	1	0	A+B′+C
0	1	1	A+B′+C′
1	0	0	A′+B+C
1	0	1	A′+B+C′
1	1	0	A′+B′+C
1	1	1	A′+B′+C′

F in canonical form:

$$F(A, B, C) = (A + B + C)\,(A + B' + C)\,(A' + B + C)$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= (A + B + C)\,(A + B' + C)\,(A' + B + C) \\ &= (A + B + C)\,(A + B' + C) \\ &\quad (A + B + C)\,(A' + B + C) \\ &= (A + C)\,(B + C) \end{aligned}$$