CSE 311: Foundations of Computing (Fall, 2015)

Homework 8 Out: Fri, 4-Dec. Due: Friday, 12-Dec, **11: 59 pm** on **Gradescope**

Additional directions: You should write down carefully argued solutions to the following problems. Your first goal is to be complete and correct. A secondary goal is to keep your answers simple and succinct. The idea is that your solution should be easy to understand for someone who has just seen the problem for the first time. (Re-read your answers with this standard in mind!) You may use any results proved in lecture (without proof). Anything else must be argued rigorously.

1. Relations warmup [12 points]

Determine whether the relation $R(A, B)$ on the set of all twitter users is reflexive, symmetric, antisymmetric, and/or transitive:

(a) Some users who have retweeted user A have also retweeted user B.
(b) There are no common followers of users A and B.
(c) There is a user that has retweeted both users A and B.
(d) Every follower of user A also follows user B.

2. Relations yoga [15 points]

Call a relation R *transflexive* if it is both transitive and reflexive.

Suppose that R and R' are transflexive relations on a set $\mathcal{A}$. Prove or disprove each of the following statements:

(a) $R \cup R'$ is transflexive

(b) $R \cap R'$ is transflexive

(c) $R \circ R'$ is transflexive

3. Symmetry and Power [20 points]

Let R be a symmetric relation on a set $\mathcal{A}$. Use induction to show that R^n is symmetric for all integers $n \geq 1$.

4. Needs more fiber [15 points]

Let $\Sigma = \{0,1,2,3,4,5,6,7,8,9, \mathrm{m}, \mathrm{o}, \mathrm{d}, =\}$.

Let L be the language of all strings of the form "$x \bmod y = z$" such that:

$x, y, z \in \{0,1,2,3,4,5,6,7,8,9\}^*$ and $y \neq 0$ and z is the remainder when x is divided by y where the strings x, y, z are interpreted as decimal. Prove that L is irregular.

5. Extra credit: The pumping lemma for context free languages

There is a "pumping lemma" for context-free languages, but it's a little complicated. If the language L has a context-free grammar, then there is some integer $p \geq 0$ such that every string $s \in L$ with $\mathrm{len}(s) \geq p$ can be written in the form $s = uvwxy$ where u, v, w, x, y are substrings satisfying:

1. $\mathrm{len}(vwx) \leq p$
2. $\mathrm{len}(vx) \geq 1$
3. $uv^n wx^n y \in L$ for all $n \geq 0$

Use this lemma to prove that the language $\{1^{m+n} 0^m 1^n 0^{m+n} \,:\, m, n \geq 0\}$ cannot be written as a context-free grammar.

5. Extra credit: Superheroes in line for the bathroom

Suppose we have a group V of n superheroes. Define a relation R on V such that $(a, b) \in R$ is true if a would defeat b in battle. The heroes all go to an outdoor concert together, but there is only one unisex bathroom. You want to arrange them in a line for the bathroom $x_1, x_2, \dots, x_n$ so that if $i < j$, then $(x_j, x_i) \notin R$. In other words, a hero $a \in V$ should not come later in the line than anyone he or she can beat up. Call a lineup **peaceful** if this conditioned is satisfied.

A **cycle of power** is a sequence $a_1, a_2, \dots, a_k \in V$ such that $(a_1, a_2), (a_2, a_3), \dots, (a_{k-1}, a_k) \in R$ and $(a_k, a_1) \in R$ (this is a cycle in the directed graph defined by R).

Use induction to prove that there is a peaceful line if and only if R does not contain any cycle of power.