CSE 311: Foundations of Computing (Autumn, 2015)

Homework 4 Out: Friday, 23-Oct. Due: Friday, 30-Oct, **11:59PM** on **Gradescope**

In this assignment, you may write all your proofs in English. Your logic still needs to be careful and correct.

1. Power sets [10 points]

Show that for any two sets S and T, it holds that:

$$\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T)$$

2. Constructing functions [18 points]

In each of the following items you need to construct a (one-to-one or onto) function with the given domain/co-domain. Give a short explanation of why your function satisfies the desired properties.

a) A one-to-one function from the positive even numbers to the positive numbers which are congruent to 2 mod 7.

b) An onto function from the positive rational numbers to positive integers (to be precise you should define $f(a, b)$ corresponding to the rational number a/b).

c) An onto function from real numbers $\{x : 0 < x \leq 1\}$ to real numbers $\{y : y \geq 1\}$.

d) An onto function from real numbers $\{x : 0 \leq x \leq 1\}$ to the positive real numbers.

e) An onto function from $\{x : 0 \leq x \leq 1\}$ to all real numbers.

f) A one-to-one function from from positive integers to prime numbers.

3. Modular Arithmetic [10 points]

Prove or disprove the following statements. You may write a computer program to gain some evidence before you try to prove the statement. If you disprove a statement, you should present a succinct counterexample (no output of a computer program!).

a) If k is a positive integer that is a multiple of 6, then $n^k \equiv 1 \pmod{7}$ for any integer n that is not a multiple of 7.

b) For every integer $n \geq 1$:

$$(n^2 + 79n + 1600)! \equiv -1 \left(\bmod\ (n^2 + 79n + 1601)\right)$$

Recall the definition of the factorial: $N! = N(N-1)(N-2)\cdots 3 \cdot 2 \cdot 1$

4. Modular Numerology [10 points]

Let a, b be integers and p, m be positive integers. Prove that if $m \mid p$ and $a \equiv b \pmod{m}$, then $a \bmod p \equiv b \bmod p \pmod{m}$.

5. Multiples of 9 [10 points]

For a positive integer n, let $s(n)$ be the sum of the digits of n in its decimal representation.

For example, if $n = 137$, then $s(n) = 1 + 3 + 7 = 11$.

Show that for any positive integer $\boldsymbol{n}$, it holds that $\boldsymbol{n \equiv s(n) \bmod 9}$.

For example, $137 \equiv 11 \bmod 9$.

6. Permuting the digits [20 points]

Let n be a positive integer. Let m be another positive integer with the property that the decimal digits of m are a permutation of the decimal digits of n. Also, assume that the leftmost digit of both m and n are nonzero.

For example, if $n = 1308$, we may have $m = 3108$, or we may have $m = 8031$, etc., but we cannot have $m = 0138$.

a) Show that $\frac{n}{m} < 10$ and $\frac{m}{n} < 10$.

b) Use part (a) and Problem 5 to show that it is never the case that both m and n are a power of 2. For example, if $n = 128 = 2^7$, none of the numbers $182, 218, 281, 812, 821$ is a power of 2.

7. [Extra Credit] Infinite Linked List

Suppose you are given a linked list and a pointer to the first element of the list.

- If x is a pointer to an element of the list, then $x.\mathrm{next}$ points to the next element of the list.
- If x points to the last element of the list, then $x.\mathrm{next} = \mathrm{null}$.

You are supposed to write a function InifiniteList(start), in Java, that returns 1 if the list ends, i.e., if there is an element x such that $x.\mathrm{next} = \mathrm{null}$ and returns 0 otherwise. Your program must return 0 if the list has a loop. Note that your program must end in a finite number of steps; in other words, you should not circle over the loop forever.

You can ONLY use two auxiliary pointers x and y to the elements of the list in the description of the function, and NO additional pointers or variables. To receive credit, you need to **prove** that your program works, i.e., it always returns the right answer in a finite number of steps.

8. [Extra Credit] Palindromes

The numbers 214412 and 278872 read the same forward and backward when written in decimal. Prove that every such number with an even number of digits is divisible by 11.