CSE 311: Foundations of Computing (Autumn, 2015)

Homework 3 Out: Friday, 16-Oct. Due: Friday, 23-Oct, **11:59PM** on **Gradescope**

1. A formal proof [10 points]

(a) [8 points] Write a formal proof that under the three assumptions:

$$p \oplus q, \qquad r \to \neg s, \qquad s \wedge p \to r,$$

the proposition $s \to q$ holds true.

(b) [2 points] How many rows would you need for a truth table proving the same statement?

2. Translate to a formal proof [10 points]

Translate the following English statement and its proof into a formal proof. Define appropriate predicates and domains of discourse. Note that you are not allowed to use quantifiers in the definition of your predicates.

For every positive integer n, if $n = a \cdot b$ then, $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

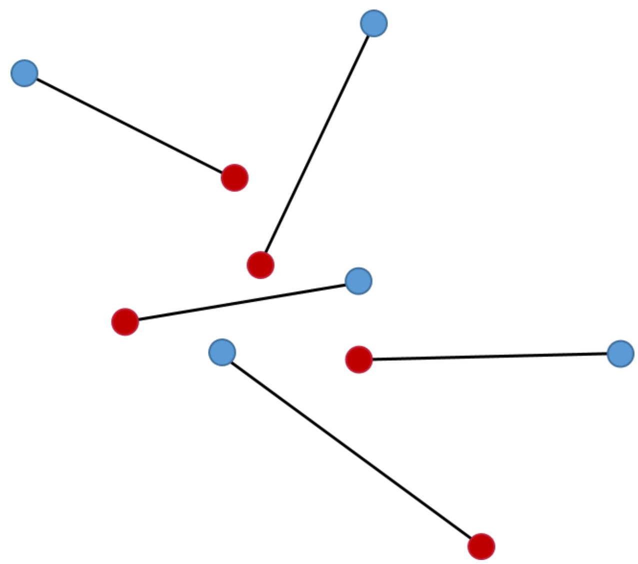
Proof: We prove the contrapositive statement. Suppose that ($a \leq \sqrt{n}$ or $b \leq \sqrt{n}$) does not hold, we show that $n \neq a \cdot b$. Since ($a \leq \sqrt{n}$ or $b \leq \sqrt{n}$) is false, it must be that $a > \sqrt{n}$ and $b > \sqrt{n}$. By simple algebra,

$$a \cdot b > \sqrt{n} \cdot b > \sqrt{n} \cdot \sqrt{n} = n.$$

Therefore, $a \cdot b \neq n$.

3. Non-crossing matchings [20 points]

In this problem, you will show that given n red points and n blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume there are n red and n blue points fixed in the plane.



(a) A *matching* M is a collection of n line segments connecting distinct red-blue pairs. The total *length* of a matching M is the sum of the lengths of the line segments in M. Say that a matching M is *minimal* if there is no matching with a smaller total length. Let $\mathrm{IsMinimal}(M)$ be the predicate that is true precisely when M is a minimal matching.

Give an (informal) argument in English explaining why there must be at least one matching M so that $\mathrm{IsMinimal}(M)$ is true. In other words: $\exists M\ \mathrm{IsMinimal}(M)$. (The domain of discourse is all matchings.)

(b) Let $\mathrm{HasCrossing}(M)$ be the predicate that is true precisely when there are two line segments in M that cross each other. Give an (informal) argument in English explaining why

$$\forall M\ \mathrm{HasCrossing}(M) \rightarrow \neg\mathrm{IsMinimal}(M)$$

You may need to draw a picture and remember some high school geometry.

(c) Now use the two results above to give a **formal** proof of the statement: $\exists M\ \neg\mathrm{HasCrossing}(M)$.

4. A faulty proof [12 points]

Theorem: Given $(p \wedge r) \rightarrow (q \wedge r)$, $\neg s \rightarrow (r \wedge q)$, and $s \rightarrow (p \wedge q)$, prove q.

"Proof:"

1.	$(p \wedge r) \rightarrow (q \wedge r)$	[Given]
2.	$\neg s \rightarrow (r \wedge q)$	[Given]
3.	$s \rightarrow (p \wedge q)$	[Given]
4.	$\neg s \rightarrow q$	[Elim $\wedge$: 2]
5.	$p \rightarrow (q \wedge r)$	[Elim $\wedge$: 1]
6.	s	[Assumption]
7.	$p \wedge q$	[MP: 6, 3]
8.	p	[Elim $\wedge$: 7]
9.	$q \wedge r$	[MP: 8, 5]
10.	q	[Elim $\wedge$: 9]
11.	$s \rightarrow q$	[Direct Proof Rule]
12.	$(s \rightarrow q) \wedge (\neg s \rightarrow q)$	[Intro $\wedge$: 4, 11]
13.	$(\neg s \vee q) \wedge (\neg\neg s \vee q)$	[Law of Implication]
14.	$(q \vee \neg s) \wedge (q \vee \neg\neg s)$	[Commutativity]
15.	$q \wedge (\neg s \vee \neg\neg s)$	[Distributivity]
16.	$q \wedge T$	[Law of Negation]
17.	q	[Domination]

(a) [3 points] Is the conclusion of the "proof" correct? Explain in words why or why not

(b) [9 points] Explain which steps of the proof are faulty and why.

5. Finding the Rabbit [37 points]

Suppose there are four boxes B_1, B_2, B_3, B_4 in a row. There is a rabbit in one of the boxes that we want to find. In each time step we open of the boxes. If the rabbit was in that box we win; otherwise we close the box and the rabbit jumps from its box to a neighboring box. For example, if it is in B_2 it will jump to either of B_1 or B_3, but if it is in B_1 it will be in B_2 in the next step. Note that the rabbit always jumps.

You are supposed to give a sequence of boxes to open and prove that by the end of the sequence we find the rabbit no matter where it started.

For example, if we only have two boxes, it is enough to open B_1 twice. If the rabbit is in B_1 at first, we find it right away. Otherwise, it will jump to B_1 and we find it in the second step.

(a) [5 points] Let $\text{Rabbit}(x, t)$ be a predicate where x has the domain of discourse is the four values {1,2,3,4} and t has the domain of nonnegative integers. $\text{Rabbit}(x, t)$ is true precisely when the rabbit is in box x at step t. For example, if the rabbit starts in box B_1, then $\text{Rabbit}(1,0) = T$ and $\text{Rabbit}(2,1) = T$. Write logical expressions (with quantifiers) to express the properties of $\text{Rabbit}(x, t)$ for all x and t based on the rules that we mentioned above.

(b) [5 points] Define the predicate $\text{FindR}(x, y, z, \ldots)$ where all the variables have domain of discourse {1,2,3,4} by

$$\text{FindR}(x, y, z, \ldots) = \text{Rabbit}(x, 0) \vee \text{Rabbit}(y, 1) \vee \text{Rabbit}(z, 2) \vee \cdots$$

[Note that formally we have a different predicate for every length of strategy.] For example, observe that $\text{Rabbit}(1,0) \to \text{FindR}(1)$. Write logical expressions (with quantifiers) using $\text{Rabbit}(x, t)$ to express $\text{FindR}(x)$ and $\text{FindR}(x, y)$ for all x and y in the domain.

(c) [5 points] Use parts (a) and (b) to give a formal proof that

$$\text{Rabbit}(3,0) \to \text{FindR}(1,2,3)$$

(d) [5 points] Give a formal proof that

$$\text{Rabbit}(1,0) \vee \text{Rabbit}(3,0) \to \text{FindR}(1,2,3)$$

(e) [7 points] Give a formal proof that

$$\forall x, y, z \left(\text{Rabbit}(2,0) \to \text{FindR}(x, y, z, 1, 2, 3)\right)$$

(f) [10 points] Give a formal proof that

$$\exists a, b, c, d, e, f\ \forall x\ (\text{Rabbit}(x, 0) \to \text{FindR}(a, b, c, d, e, f)).$$

6. Logic and Calculus [Extra Credit]

The following is a proof that $\frac{d}{dx}x^3$ at 1 is equal to 3. You are supposed to translate this proof into logic. Define appropriate predicates and domains of discourse for the variables. Note that you are not allowed to use quantifiers in the definition of your predicates.

By the definition of the derivative we need to show the following: For any number $0 < \epsilon < 1$, there is $\delta > 0$ such that for any x where $|x| < \delta$,

$$\left|\frac{(1+x)^3 - 1^3}{x}\right| \leq 3 + \epsilon.$$

Let $\epsilon > 0$. Set $\delta = \epsilon/4$. We have

$$\left|\frac{(1+x)^3 - 1^3}{x}\right| = \left|\frac{x^3 + 3x^2 + 3x}{x}\right| = |x^2 + 3x + 3| \leq \delta^2 + 3\delta + 3 = 3 + \frac{\epsilon^2 + 12\epsilon}{16} \leq 3 + \frac{13\epsilon}{16} \leq 3 + \epsilon.$$

The first inequality uses $|x| < \delta$ and the last equality uses $\delta = \frac{\epsilon}{4}$ and the second to last inequality uses $\epsilon < 1$.