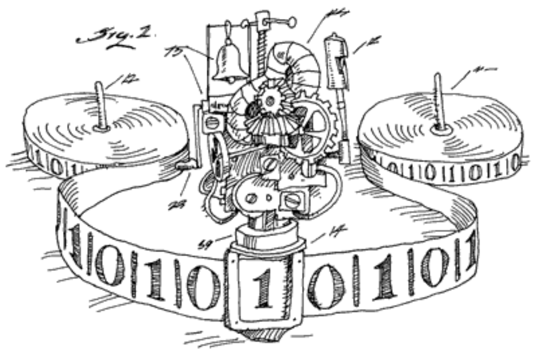
CSE 311: Foundations of Computing

Fall 2014

Lecture 28: Undecidability



Review:

- We have seen that
  - The set of all (Java) programs is countable
  - The set of all functions f : $\mathbb{N}_+ \to \{0,1,\ldots,9\}$ is not countable
- Let's review that second proof
  - Consider any listing of such functions

    $f_1, f_2, f_3, f_4, \ldots$

Supposed listing of functions: $\mathbb{N}_+ \to \{0,1,\ldots,9\}$

	1	2	3	4	5	6	7	8	9	...
f_1	5	0	0	0	0	0	0	0	...	...
f_2	3	3	3	3	3	3	3	3	...	...
f_3	1	4	2	8	5	7	1	4	...	...
f_4	1	4	1	5	9	2	6	5	...	...
f_5	1	2	1	2	2	1	2	2	...	...
f_6	2	5	0	0	0	0	0	0	...	...
f_7	7	1	8	2	8	1	8	2	...	...
f_8	6	1	8	0	3	3	9	4	...	...
...	...	....	....	...	...	...	...	...	...	

Flipped Diagonal Function D: $\mathbb{N}_+ \to \{0,1,\ldots,9\}$

Flipping Rule:

If $f_n(n)$ is 5, make $D(n)=1$

If $f_n(n)$ is not 5, make $D(n)=5$

	1	2	3	4						
f_1	5^{1}	0	0	0						
f_2	3	3^{5}	3	3						
f_3	1	4	2^{5}	8	5	7	1	4	...	...
f_4	1	4	1	5^{1}	9	2	6	5	...	...
f_5	1	2	1	2	2^{5}	1	2	2	...	...
f_6	2	5	0	0	0	0^{5}	0	0	...	...
f_7	7	1	8	2	8	1	8^{5}	2	...	...
f_8	6	1	8	0	3	3	9	4^{5}	...	...
...	...	....	....	...	...	...	...	...	......	

Flipped Diagonal Function D: $\mathbb{N}_+ \to \{0,1,\ldots,9\}$

	1	2	3	4	5	6	7	8	9	...
D =	1									
		5								
			5							
				1						
					5					
						5				
							5			
								5		
									...	

For all n, we have
$D \neq f_n$ since $D(n) \neq f_n(n)$

$\Rightarrow$ list was incomplete

$\Rightarrow \{f \mid f : \mathbb{N}_+ \to \{0,1,\ldots,9\}\}$ is not countable

Non-computable functions

- We have seen that
  - The set of all (Java) programs is countable
  - The set of all functions $f : \mathbb{N}_+ \to \{0,1,\ldots,9\}$ is not countable
- So...
  - There must be some function $f : \mathbb{N}_+ \to \{0,1,\ldots,9\}$ that is not computable by any program!

Back to the Halting Problem

- Suppose that there is a program **H** that computes the answer to the Halting Problem
- We will build a table with a row for each program (just like we did for uncountability of reals)
- If the supposed program **H** exists then the **D** program we constructed as before will exist and so have a row in the table
- We will see that **D** must have entries like the "flipped diagonal"
  - **D** can't possibly be in the table.
  - Only assumption was that **H** exists. That must be false.

Some possible inputs **x**

<P> is shorthand for CODE(P)

programs **P**

	$<P_1>$	$<P_2>$	$<P_3>$	$<P_4>$	$<P_5>$	$<P_6>$	....					
P_1	0	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1	0	1	0	...
P_9	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
.												

(P,x) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

Some possible inputs **x**

programs **P**

	$<P_1>$	$<P_2>$	$<P_3>$	$<P_4>$	$<P_5>$	$<P_6>$	....					
P_1	0	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1	0	1	0	...
P_9	.	. . . .	.	.	.	.	.	.	.			
.	.	. . . .	.	.	.	.	.	.	.			
.												

(**P**,**x**) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

recall: code for D assuming subroutine H that solves the halting problem

- Function **D(x)**:
  - if **H(x,x)==true** then
    - **while** (true); /* loop forever */
  - else
    - **return**; /* do nothing and halt */
  - endif

Some possible inputs **x**

D behaves like **flipped diagonal**

programs **P**

	$<P_1>$	$<P_2>$	$<P_3>$	$<P_4>$	$<P_5>$	$<P_6>$	....					
P_1	0 **1**	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1 **0**	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1 **0**	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0 **1**	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1 **0**	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1 **0**	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0 **1**	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1 **0**	0	1	0	...
P_9	.	. . . .	.	.	.	.	.	.	.			
.	.	. . . .	.	.	.	.	.	.	.			
.												

(**P**,**x**) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

recall: code for D assuming subroutine H that solves the halting problem

- Function **D(x)**:
  - if **H(x,x)==true** then
    - **while** (true); /* loop forever */
  - else
    - **return**; /* do nothing and halt */
  - endif
- If **D** existed it would have a row different from every row of the table
  - **D** can't be a program so **H** cannot exist!

Diagram: Using Hypothetical Program H to build D

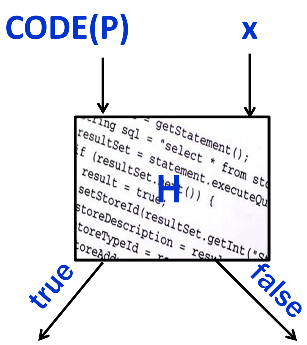
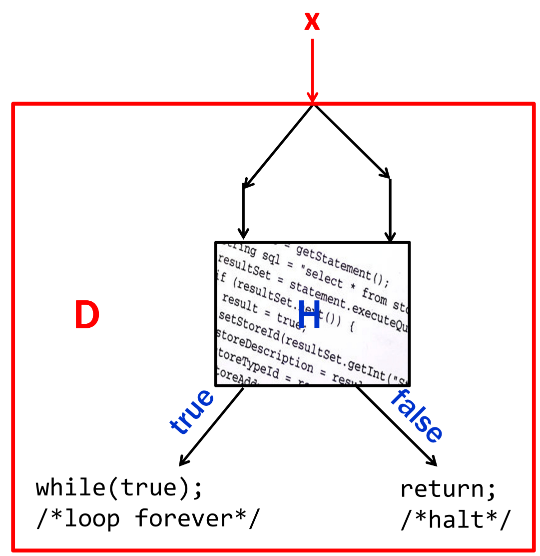


Diagram: Using Hypothetical Program H to build D



More than just halting is hard

- We showed
  - if the hypothetical program H deciding the Halting Problem existed, then we could use it to build a program D that cannot possibly exist
  - Since D doesn't exist, program H cannot exist
- We will use similar approach to show that other important problems are hard
  - if there is a hypothetical program A solving one of these problems, then we could use it to build a program H solving the Halting Problem
  - Since H doesn't exist, A cannot exist

But first another hard halting-related problem

Halting Problem:

Given: - CODE(**P**) for any program **P**
- input **x**

Output: **true** if **P** halts on input **x**
false if **P** does not halt on input **x**

HaltsNoInput Problem:

Given: - CODE(**Q**) for any program **Q**

Output: **true** if **Q** halts without reading any input
false if **Q** reads input or runs forever without reading any input.

Key idea: Hardcoding an Input

```
INPUT is “potato”

public String P(String y) {
    return new String(
        Arrays.sort(y.toCharArray()
    );
}
```

Q: Version of P with “hardcoded” input:

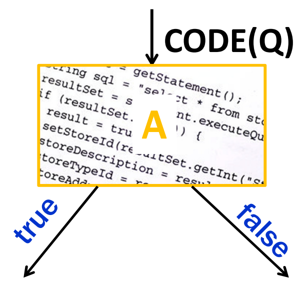
```
public String Q() {
    return new String(
        Arrays.sort(“potato”.toCharArray()
    );
}
```

Q() behaves the same as P(“potato”), except that it doesn’t read any input.

Can write a program `Hardcoder` that, given CODE(P) and an input string x, produces CODE(Q)

Showing there is no program solving **HaltsNoInput**

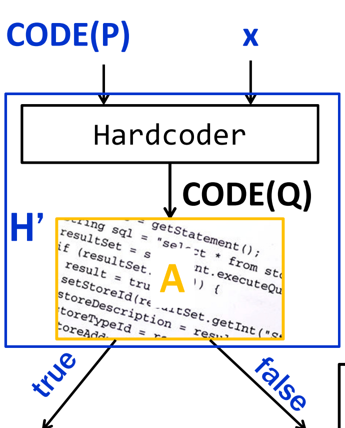
Suppose that hypothetical program A solves HaltsNoInput problem.



A outputs true
iff Q() reads no input
and (always) halts

Showing there is no program solving **HaltsNoInput**

Suppose that hypothetical program A solves HaltsNoInput problem. ***Combine with*** `Hardcoder`:



H’ outputs true on inputs CODE(P) and x
***iff* A outputs true**
iff Q() reads no input and (always) halts
iff* P(x) *halts

If A existed then H’ would solve the Halting Problem: Impossible

Some notation: Decision problems as sets

Every decision problem can be written as asking about membership in a set.

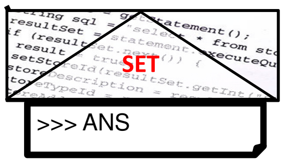
If a program “decides” a set, then it must output true on all inputs in the set and false on all inputs not in the set.

Halt ={(CODE(P),x) : P is a program that halts on input **x**}

HaltsNoInput={CODE(Q): Q is a program that halts without reading any input}

Convenient pictures

- Rather than continue to come up with more names like H and A for our hypothetical programs…
- Given a decision problem SET we use the following picture to denote any hypothetical program that solves decision problem SET with ANS denoting its output.



Showing HELLO is Undecidable

Consider the set:

HELLO = {CODE(R) : R is a program that reads no input, prints "Hello", and always halts}

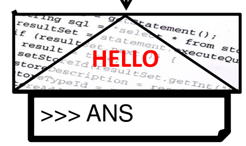
CODE(Q)

Question: Does Q() halt?

Step 1: Remove all `System.out.print/println` statements from CODE(Q).
Step 2: Append `System.out.println("Hello")` at the end of the program code.
Call the new program R

CODE(R)

Question: Does **R**() print "**Hello**" and halt?



Answering with ANS **would solve HaltsNoInput!**

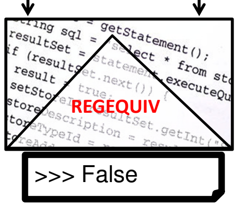
A Decision Problem We Can Solve

REGEQUIV = {(R_1, R_2) : R_1 and R_2 are equivalent regexps}

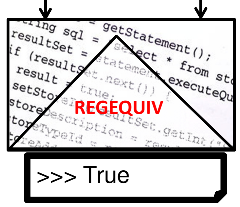
In this case the hypothetical program does exist:

Convert both to NFAs then DFAs, minimize and compare

00(10)*11** *0*(01)*1**



00(10)*11** *0*(01)*01*1*



Showing EQUIV is Undecidable

Consider the set:

EQUIV = {(CODE(P), CODE(R)): P, R are programs, P(x) = R(x) for all inputs x}

CODE(Q)

Question: Does Q() halt?

Step 1: Construct P:

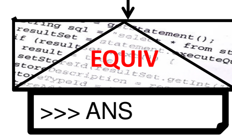
```
public static boolean P() {return true;}
```

Step 2: Construct R:
- Step a: Replace return type of Q with boolean
- Step b: Replace all return values with true
- Step c: Add "return true;" to the end of the program

Call this program R

(CODE(P),CODE(R))

Question: Are P and R Equivalent?



Answering with ANS **would solve HaltsNoInput!**

Pitfalls

- **Not *every* problem on programs is undecidable! Which of these is decidable?**
- Input `CODE(P)` and x
  Output: true if P prints "ERROR" on input x
  after less than 100 steps
  false otherwise

- Input `CODE(P)` and x
  Output: true if P prints "ERROR" on input x
  after more than 100 steps
  false otherwise