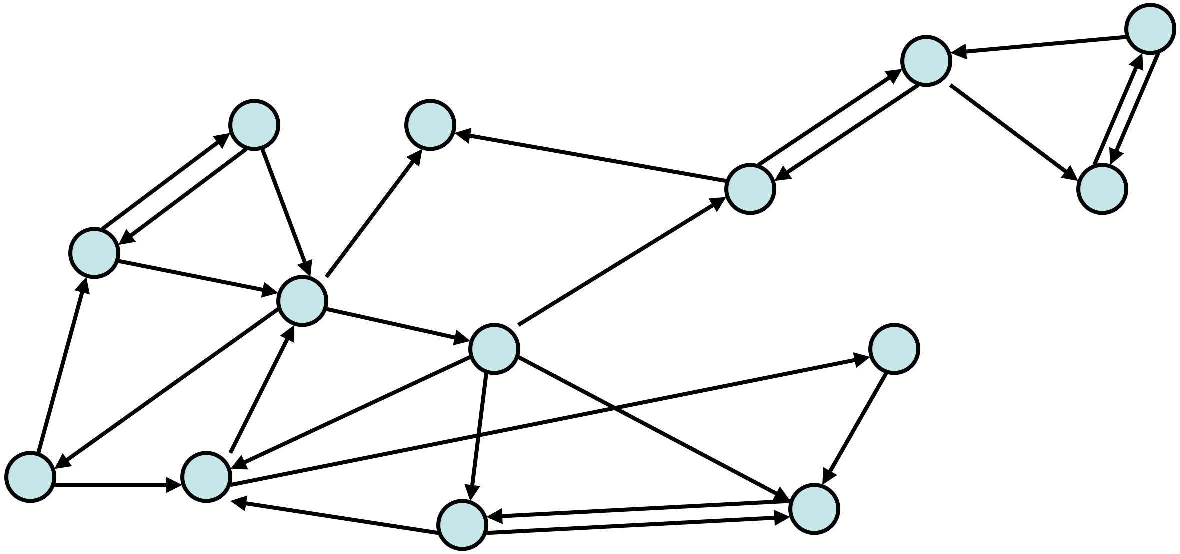
CSE 311: Foundations of Computing

Fall 2014

Lecture 20: Relations and Directed Graphs



Relations

Let A and B be sets,
A binary relation from A to B is a subset of $A \times B$

Let A be a set,
A binary relation on A is a subset of $A \times A$

Relations You Already Know!

$\geq$ on $\mathbb{N}$

That is: $\{(x,y) : x \geq y \text{ and } x, y \in \mathbb{N}\}$

$<$ on $\mathbb{R}$

That is: $\{(x,y) : x < y \text{ and } x, y \in \mathbb{R}\}$

$=$ on Σ^*

That is: $\{(x,y) : x = y \text{ and } x, y \in \Sigma^*\}$

$\subseteq$ on P(U) for universe U

That is: $\{(A,B) : A \subseteq B \text{ and } A, B \in P(U)\}$

Relation Examples

$R_1 = \{(a, 1), (a, 2), (b, 1), (b, 3), (c, 3)\}$

$R_2 = \{(x, y) \mid x \equiv y \pmod 5\}$

$R_3 = \{(c_1, c_2) \mid c_1$ is a prerequisite of $c_2\}$

$R_4 = \{(s, c) \mid$ student s had taken course c $\}$

Properties of Relations

Let R be a relation on A.

R is reflexive iff $(a,a) \in R$ for every $a \in A$

R is symmetric iff $(a,b) \in R$ implies $(b, a) \in R$

R is antisymmetric iff $(a,b) \in R$ and $a \neq b$ implies $(b,a) \notin R$

R is transitive iff $(a,b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

Combining Relations

Let R be a relation from A to B.
Let S be a relation from B to C.

The composition of R and S, $S \circ R$ is the relation from A to C defined by:

$$S \circ R = \{(a, c) \mid \exists\, b \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$$

Intuitively, a pair is in the composition if there is a "connection" from the first to the second.

Examples

(a,b) ∈ Parent iff b is a parent of a

(a,b) ∈ Sister iff b is a sister of a

When is (x,y) ∈ Sister Parent?

When there is a person z, where (x, z) ∈ Parent and (z, y) ∈ Sister. That is, z is a parent of x, and y and z are sisters. Or, y is x's Aunt.

When is (x,y) ∈ Parent Sister?

When there is a person z, where (x, z) ∈ Sister and (z, y) ∈ Parent. That is, z and x are sisters, and y is a parent of z. Or, y is x's parent.

S ∘ R = {(a, c) | ∃ b such that (a,b)∈ R and (b,c)∈ S}

Examples

Using the relations: Parent, Child, Brother, Sister, Sibling, Father, Mother, Husband, Wife express:

Uncle: b is an uncle of a

$(a,b) \in (\text{Brother} \circ \text{Parent} \cup \text{Husband} \circ \text{Sibling} \circ \text{Parent})$

Cousin: b is a cousin of a

$(a,b) \in (\text{Child} \circ \text{Sibling} \circ \text{Parent})$

Powers of a Relation

Let R be a relation on A.

$R^2 = R \circ R = \{(a, c) : \exists b\ ((a,b) \in R \text{ and } (b, c) \in R$

$R^0 = \{(a, c) : a \ \in A\} = A$

$R^1 = \{(a, b) : (a, b) \in R\} = R$

$R^{n+1} = R^n \circ R$

Matrix Representation

Let R be a relation on A={a_1, a_2, ..., a_p}.

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, a_j) \in R, \\ 0 \text{ if } (a_i, a_j) \notin R. \end{cases}$$

{(1, 1), (1, 2), (1, 4), (2,1), (2,3), (3,2), (3, 3), (4,2), (4,3)}

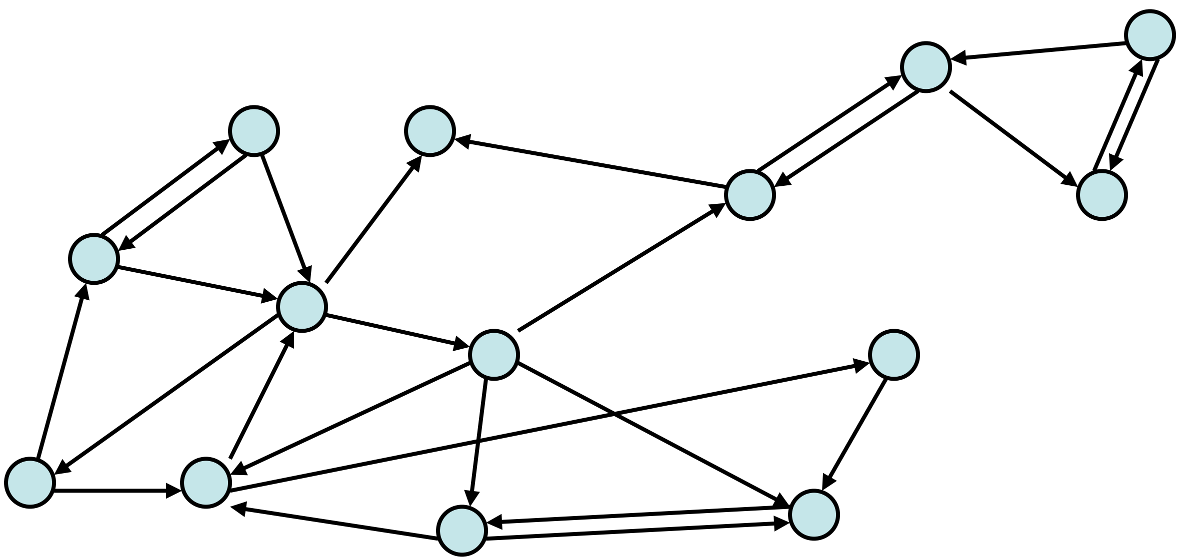
	1	2	3	4
1	1	1	0	1
2	1	0	1	0
3	0	1	1	0
4	0	1	1	0

Directed Graphs

G = (V, E) V – vertices
E – edges, ordered pairs of vertices

Path: $v_0, v_1, \ldots, v_k$, with (v_i, v_{i+1}) in E

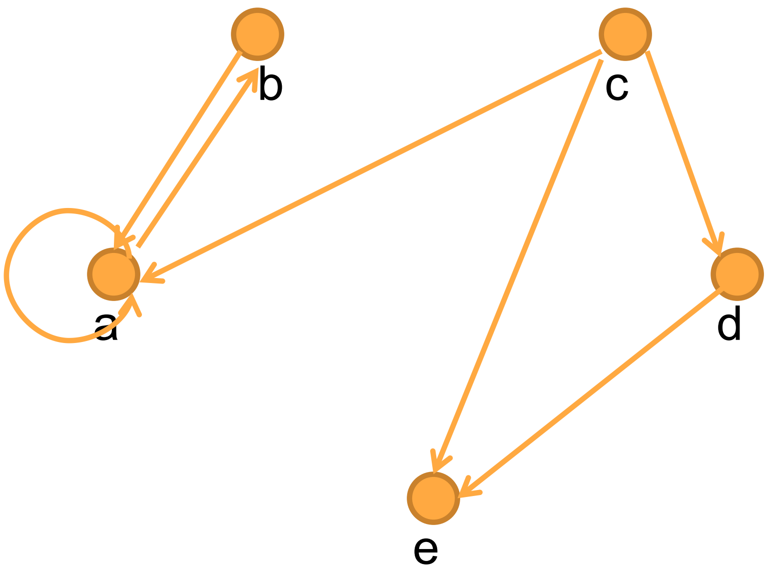
Simple Path
Cycle
Simple Cycle



Representation of Relations

Directed Graph Representation (Digraph)

{(a, b), (a, a), (b, a), (c, a), (c, d), (c, e) (d, e) }



Relational Composition using Digraphs

If $S=\{(2,2),(2,3),(3,1)\}$ and $R=\{(1,2),(2,1),(1,3)\}$

Compute $S \circ R$

(1, 2) ∈ R and (2, 2) ∈ S means (1, 2) ∈ S ∘ R
(1, 2) ∈ R and (2, 3) ∈ S means (1, 3) ∈ S ∘ R
(2, 1) ∈ R and (2, 2) ∈ S means (2, 2) ∈ S ∘ R

Paths in Relations and Graphs

A path in a graph of length n is a list of edges with vertices next to each other.

Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$

Connectivity In Graphs

Two vertices in a graph are connected iff there is a path between them.

Let R be a relation on a set A. The connectivity relation R* consists of the pairs (a,b) such that there is a path from a to b in R.

$$R^* = \bigcup_{k=0}^{\infty} R^k$$

Note: The text uses the wrong definition of this quantity. What the text defines (ignoring k=0) is usually called R^+

Properties of Relations (again)

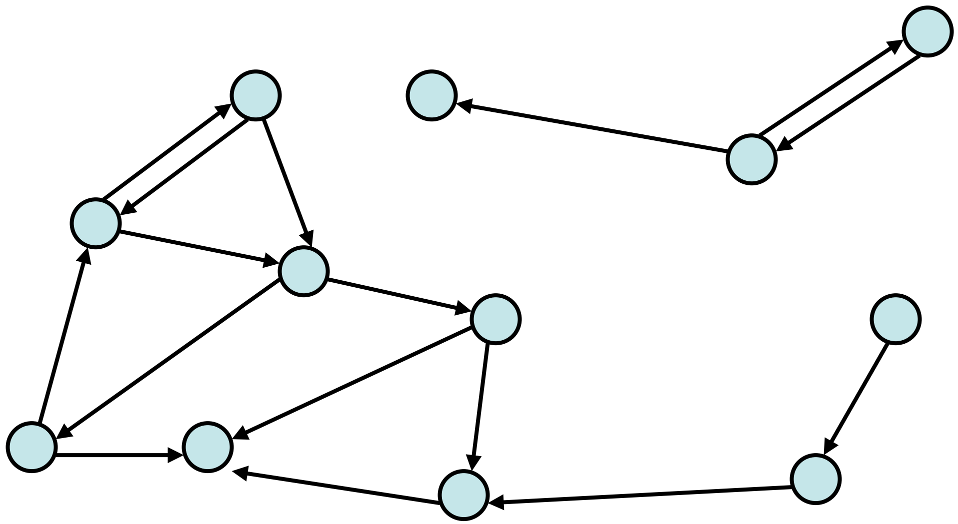
Let R be a relation on A.

R is reflexive iff $(a,a) \in R$ for every $a \in A$

R is symmetric iff $(a,b) \in R$ implies $(b, a) \in R$

R is transitive iff $(a,b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

Transitive-Reflexive Closure



Add the minimum possible number of edges to make the relation transitive and reflexive.

The transitive-reflexive closure of a relation R is the connectivity relation R*