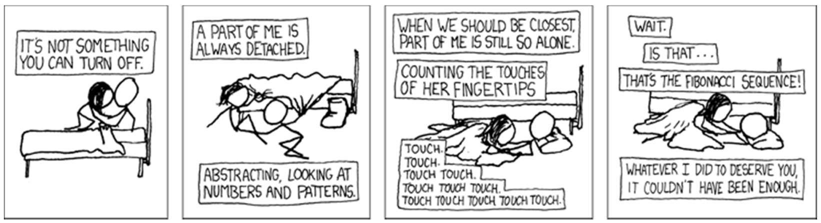
CSE 311: Foundations of Computing

Fall 2014

Lecture 17: Recursive Definitions and Structural Induction



Administrivia

Midterm

Monday, November 3rd, IN CLASS

Topics: Everything up to ordinary induction.

Review sessions:

~~**Thursday, Oct 30: 4:30pm – 6:00pm (THO 101)**~~

Saturday, Nov 1: 4:30pm – 7:00pm (EEB 125)

Sunday, Nov 2: 4:00pm – 6:00pm (EEB 125)

Strings

- **An *alphabet* Σ is any finite set of characters**
- **The set Σ^* of *strings* over the alphabet Σ is defined by**
  - **Basis:** $\varepsilon \in \Sigma^*$ **(ε is the empty string)**
  - **Recursive: if** $w \in \Sigma^*$, $a \in \Sigma$, **then** $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards (e.g. "abba", "tht", "neveroddoreven").

Basis

- $\mathcal{E}$ is a palindrome
- **a** is a palindrome for every **a** $\in \Sigma$

Recursive Step

- If **p** is a palindrome and **a** $\in \Sigma$, then **apa** is a palindrome.

All Binary Strings with no 1's before 0's…

Write a recursive definition for the set of binary strings in which all 0's appear before any 1's in the entire string.

Function Definitions on Recursively Defined Sets

Length:

$len(\varepsilon) = 0$

$len(wa) = 1 + len(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

Number of c's in a string:

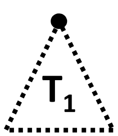
$\#_c(\varepsilon) = 0$

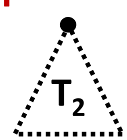
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

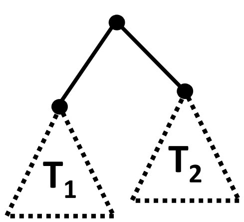
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

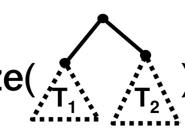
If 

and 

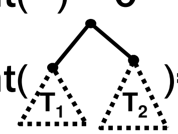
are rooted binary trees,

then so is: 

Functions Defined on Rooted Binary Trees

- size(•) = 1
- size(

) = 1 + size(T_1) + size(T_2)

- height(•) = 0
- height(

)=1 + max{height(T_1), height(T_2)}

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive Step: If $k \in \mathbb{N}$ then $k+1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Let $Q(n)$ be true iff for all $x \in S$ that take n recursive steps to be constructed, $P(x)$ is true.

Using Structural Induction

- Let S be given by…
  - **Basis:** $6 \in S$; $15 \in S$;
  - **Recursive:** if $x, y \in S$ then $x + y \in S$.

Claim: Every element of S is divisible by 3.

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Structural Induction for Strings

Let S be a set of strings over $\{a,b\}$ defined as follows…

Basis: $a \in S$

Recursive:

If $w \in S$ then $aw \in S$ and $baw \in S$

If $u \in S$ and $v \in S$ then $uv \in S$

Claim: **If $w \in S$, then w has more a's than b's.**

Claim: If $w \in S$, then $\#_a(w) > \#_b(w)$

Basis: $a \in S$

Recursive:

If $w \in S$ then $aw \in S$ and $baw \in S$

If $u \in S$ and $v \in S$ then $uv \in S$

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$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

Claim: len(x • y) = len(x) + len(y) for all x,y $\in \Sigma^*$

Let P(y) be "$len(x \bullet y) = len(x) + len(y)$ for all $x \in \Sigma^*$"

Functions Defined on Rooted Binary Trees

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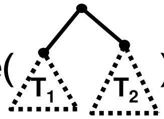
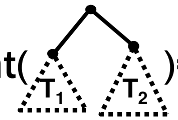
- size(

) = 1 + size(T_1) + size(T_2)

- height(•) = 0
- height(

)=1 + max{height(T_1), height(T_2)}

Claim: For every rooted binary tree T, size(T) ≤ $2^{\text{height}(T) + 1} - 1$