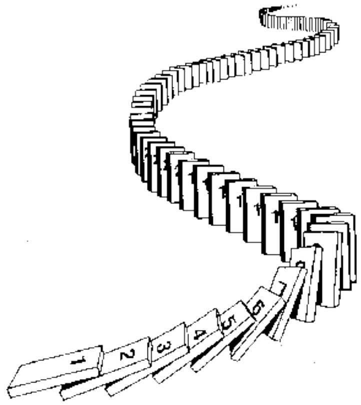
CSE 311: Foundations of Computing

Fall 2014

Lecture 15: Strong Induction



Administrivia

- Midterm is in a week in lecture!
  - We will put out a practice exam + practice questions later today!
  - There will be three review sessions (one on Thursday, one on Saturday, and one on Sunday)

Prove $3^n \geq n^2$ for all $n \geq 3$.

Let P(n) be "$3^n \geq n^2$" for all $n \geq 3$.

We go by induction on n.

Base Case:

$3^3 = 27 \geq 9 = 3^2$. So, P(3) is true.

Induction Hypothesis:

Suppose P(k) is true for some arbitrary $k \geq 3$.

Induction Step:

Note that $3^{k+1} = 3(3^k) \geq 3(k^2)$, by the IH.

Furthermore, note that $(k+1)^2 = k^2 + 2k + 1$.

Note that since $k \geq 3$, $k^2 \geq 3k \geq 2k$. And similarly, $k^2 \geq 1$.

So, continuing from above:

$$3^{k+1} = 3(3^k) \geq 3(k^2) = k^2 + k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$$

Since this is exactly P(k+1), we've shown P(k) → P(k+1)

Thus, P(n) is true for all $n \geq 3$, by induction.

Prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Note that $2(n+1)^3 = 2n^3 + 6n^2 + 6n + 2$.

Let P(n) be "$\mathbf{2n^3 + 2n - 5 \geq n^2}$" for all $n \geq 2$.

We go by induction on n.

Base Case:

$2*2^3 + 2*2 - 5 = 45 \geq 4 = 2^2$. So, P(0) is true.

Induction Hypothesis:

Suppose P(n) is true for some arbitrary $n \geq 2$.

Induction Step: Then, note that…

$(n+1)^2$	$\leq$	$n^2 + 2n + 1$	
	$\leq$	$(2n^3 + 2n - 5) + 2n + 1$	(by IH)
	$\leq$	$(2n^3 + 4n + 1) - 5$	(Re-arranging)
	$\leq$	$(2n^3 + 6n^2 + 6n + 2) - 5$	($4n + 1 \leq 6n + 6n^2 + 2$)
	$\leq$	$2(n+1)^3 - 5$	(Factoring)
	$\leq$	$2(n+1)^3 + 2n - 5$	($0 \leq 2n$)

Since this is exactly P(k+1), we've shown P(k) → P(k+1)

Thus, P(n) is true for all $n \geq 3$, by induction.

Strong Induction

$P(0)$
$\forall k\ \big((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)\big)$

$\therefore \forall n\ P(n)$

Follows from ordinary induction applied to
$Q(n) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(n)$

Strong Induction English Proof

1. **By induction we will show that $P(n)$ is true for every $n \geq 0$**
2. **Base Case: Prove $P(0)$**
3. **Inductive Hypothesis:**
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. **Inductive Step:**
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. **Conclusion: Result follows by induction**

Every integer at least 2 is the product of primes

We go by strong induction. Let P(n) be "n can be expressed as a product of primes" for n ≥ 2.

Base Case:

Note that 2 is prime; so, we can express it as "2" which is a product of primes.

Induction Hypothesis:

Suppose P(2) ∧ P(3) ∧ ⋅⋅⋅ ∧ P(k) is true for some k ≥ 2.

Induction Step:

We go by cases.
Suppose k+1 is prime. Then, "k+1" is a product of primes.
Suppose k+1 is composite. Then, k+1 = ab for some a and b such that 1 < a, b < k+1.
By our IH, we know a = $p_1p_2\cdots p_m$ and b = $q_1q_2\cdots q_n$.
So, k+1 = ab = "$p_1p_2\cdots p_m q_1q_2\cdots q_n$", which is a product of primes.

Thus, our claim is true for n ≥ 2 by strong induction.

Recursive Definitions of Functions

- $F(0) = 0;\ F(n+1) = F(n) + 1$ **for all** $n \geq 0$
- $G(0) = 1;\ G(n+1) = 2 \times G(n)$ **for all** $n \geq 0$
- $0! = 1;\ (n+1)! = (n+1) \times n!$ **for all** $n \geq 0$
- $H(0) = 1;\ H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$

Fibonacci Numbers

$f_0 = 0$
$f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding the Fibonacci Numbers

Theorem: $f_n < 2^n$ for all $n \geq 2$.