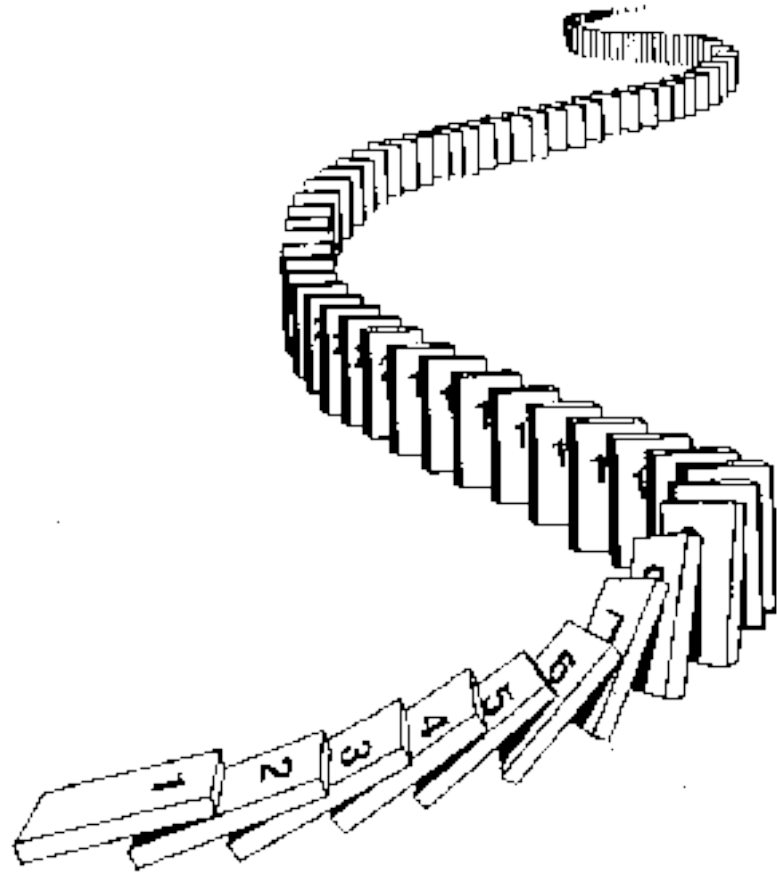


CSE 311: Foundations of Computing

Fall 2014

Lecture 15: Strong Induction



Administrivia

- **Midterm is in a week in lecture!**
 - We will put out a practice exam + practice questions later today!
 - There will be three review sessions (one on Thursday, one on Saturday, and one on Sunday)

Prove $3^n \geq n^2$ for all $n \geq 3$.

Let $P(n)$ be “ $3^n \geq n^2$ ” for all $n \geq 3$.

We go by induction on n .

Base Case:

$3^3 = 27 \geq 9 = 3^2$. So, $P(3)$ is true.

Induction Hypothesis:

Suppose $P(k)$ is true for some arbitrary $k \geq 3$.

Induction Step:

Note that $3^{k+1} = 3(3^k) \geq 3(k^2)$, by the IH.

Furthermore, note that $(k+1)^2 = k^2 + 2k + 1$.

Note that since $k \geq 3$, $k^2 \geq 3k \geq 2k$. And similarly, $k^2 \geq 1$.

So, continuing from above:

$$3^{k+1} = 3(3^k) \geq 3(k^2) = k^2 + k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$$

Since this is exactly $P(k+1)$, we've shown $P(k) \rightarrow P(k+1)$

Thus, $P(n)$ is true for all $n \geq 3$, by induction.

Prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Note that $2(n+1)^3 = 2n^3 + 6n^2 + 6n + 2$.

Let $P(n)$ be “ $2n^3 + 2n - 5 \geq n^2$ ” for all $n \geq 2$.

We go by induction on n .

Base Case:

$$2 \cdot 2^3 + 2 \cdot 2 - 5 = 45 \geq 4 = 2^2. \text{ So, } P(2) \text{ is true.}$$

Induction Hypothesis:

Suppose $P(n)$ is true for some arbitrary $n \geq 2$.

Induction Step: Then, note that...

$$\begin{aligned} (n+1)^2 &\leq n^2 + 2n + 1 \\ &\leq (2n^3 + 2n - 5) + 2n + 1 && \text{(by IH)} \\ &\leq (2n^3 + 4n + 1) - 5 && \text{(Re-arranging)} \\ &\leq (2n^3 + 6n^2 + 6n + 2) - 5 && (4n + 1 \leq 6n + 6n^2 + 2) \\ &\leq 2(n+1)^3 - 5 && \text{(Factoring)} \\ &\leq 2(n+1)^3 + 2n - 5 && (0 \leq 2n) \end{aligned}$$

Since this is exactly $P(k+1)$, we've shown $P(k) \rightarrow P(k+1)$

Thus, $P(n)$ is true for all $n \geq 2$, by induction.

Strong Induction

$$P(0)$$

$$\forall k \left((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k + 1) \right)$$

$$\therefore \forall n P(n)$$

Follows from ordinary induction applied to

$$Q(n) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(n)$$

Strong Induction English Proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
Prove that $P(k + 1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

Every integer at least 2 is the product of primes

We go by strong induction. Let $P(n)$ be “ n can be expressed as a product of primes” for $n \geq 2$.

Base Case:

Note that 2 is prime; so, we can express it as “2” which is a product of primes.

Induction Hypothesis:

Suppose $P(2) \wedge P(3) \wedge \dots \wedge P(k)$ is true for some $k \geq 2$.

Induction Step:

We go by cases.

Suppose $k+1$ is prime. Then, “ $k+1$ ” is a product of primes.

Suppose $k+1$ is composite. Then, $k+1 = ab$ for some a and b such that $1 < a, b < k+1$.

By our IH, we know $a = p_1 p_2 \dots p_m$ and $b = q_1 q_2 \dots q_n$.

So, $k+1 = ab = “p_1 p_2 \dots p_m q_1 q_2 \dots q_n”$, which is a product of primes.

Thus, our claim is true for $n \geq 2$ by strong induction.

Recursive Definitions of Functions

- $F(0) = 0; F(n + 1) = F(n) + 1$ for all $n \geq 0$
- $G(0) = 1; G(n + 1) = 2 \times G(n)$ for all $n \geq 0$
- $0! = 1; (n + 1)! = (n + 1) \times n!$ for all $n \geq 0$
- $H(0) = 1; H(n + 1) = 2^{H(n)}$ for all $n \geq 0$

Fibonacci Numbers

$$f_0 = 0$$

$$f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding the Fibonacci Numbers

Theorem: $f_n < 2^n$ for all $n \geq 2$.