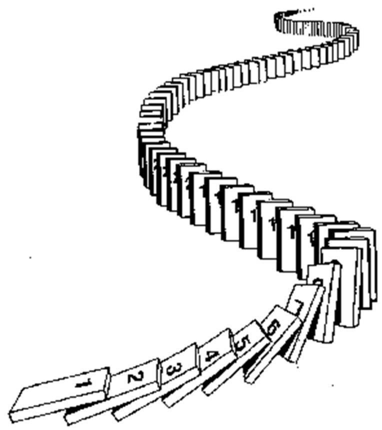
CSE 311: Foundations of Computing

Fall 2014
Lecture 14: Induction



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- **Particularly useful for reasoning about programs!**

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x – 1)+1; }
}
```

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

Prove for all n > 0, a is odd → a^n is odd

Let n > 0 be arbitrary.

Suppose that a is odd. We know that if a, b are odd, then ab is also odd.

So,

$$(\ldots\bullet ((a\bullet a)\bullet a)\bullet\ldots\bullet a) = a^n$$

(n times)

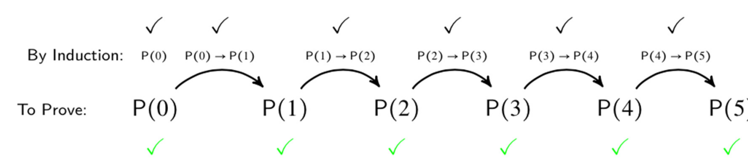
Those "…"s are a problem! We're trying to say "we can use the same argument over and over"… We'll come back to this.

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\frac{P(0) \qquad \forall\, k\, (P(k) \rightarrow P(k+1))}{\therefore\ \forall\, n\, P(n)}$$

How does this technique prove P(5)?



First, we prove P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \,\rightarrow\, P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

Instead, Let's Use Induction

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \,\rightarrow\, P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

Base Case
1. Prove P(0)

Inductive Hypothesis
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true

Inductive Step
    4. ...
    5. Prove P(k+1) is true

Conclusion
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

5 Steps To Inductive Proofs In English

Proof:

1. "We will show that P(n) is true for every n ≥ 0 by Induction."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"
   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !!)
5. "Conclusion: Result follows by induction"

What can we say about $1 + 2 + 4 + 8 + \ldots + 2^n$

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

- Can we describe the pattern?
  - $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Proving $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

- We could try proving it normally…
  - We want to show that $1 + 2 + 4 + \ldots + 2^n = 2^{n+1}$.
  - So, what do we do now? We can sort of explain the pattern, but that's not a proof…
- We could prove it for n=1, n=2, n=3, …(individually), but that would literally take forever…

5 Steps To Inductive Proofs In English

Proof:

1. "We will show that P(n) is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"
   Assume P(k) is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Want to prove that P(k+1) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !!)
5. "Conclusion: Result follows by induction"

Proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

Proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$

1. Let P(n) be "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". We will show P(n) is true for all natural numbers by induction.
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$
3. **Induction Hypothesis:** Suppose that P(k) is true for some arbitrary $k \geq 0$.
4. **Induction Step:**
   Goal: Show P(k+1), i.e. show $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$
   $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ by IH
   Adding 2^{k+1} to both sides, we get:
   $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$
   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.
   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly P(k+1).
5. Thus P(k) is true for all $k \in \mathbb{N}$, by induction.

Another Example

We want to prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

Finding A Pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- …

Prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

Prove: For all $n \geq 1$: $1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Prove: $n^n \geq n!$ for all $n \geq 1$.

- Recall: n! is the product of all the integers between 1 and n.

Checkerboard Tiling

Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: