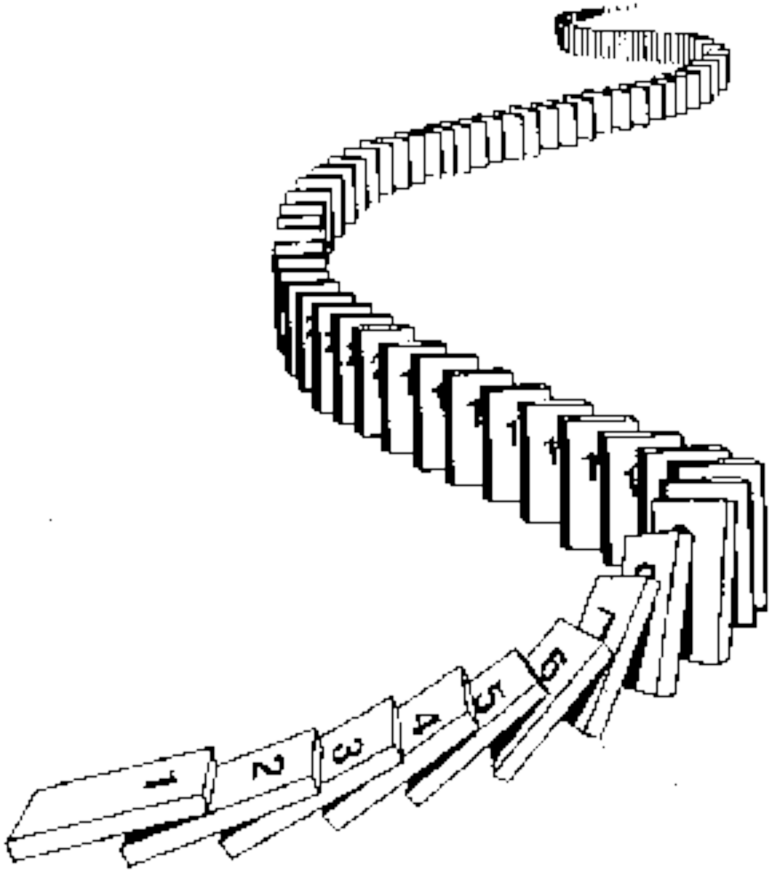
CSE 311: Foundations of Computing

Fall 2014

Lecture 14: Induction



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- **Particularly useful for reasoning about programs!**

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x – 1); }
}
```

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

Prove for all $k > 0$, n^k even $\rightarrow$ n even

Let $k > 0$ be arbitrary. We go by contrapositive. Suppose that n is odd. We know that if a, b are odd, then ab is also odd.

So,

$$(\ldots\bullet ((n\bullet n) \bullet n) \bullet\ldots\bullet n) = n^k$$

(k times)

Those "…"s are a problem! We're trying to say "we can use the same argument over and over"… We should use induction instead.

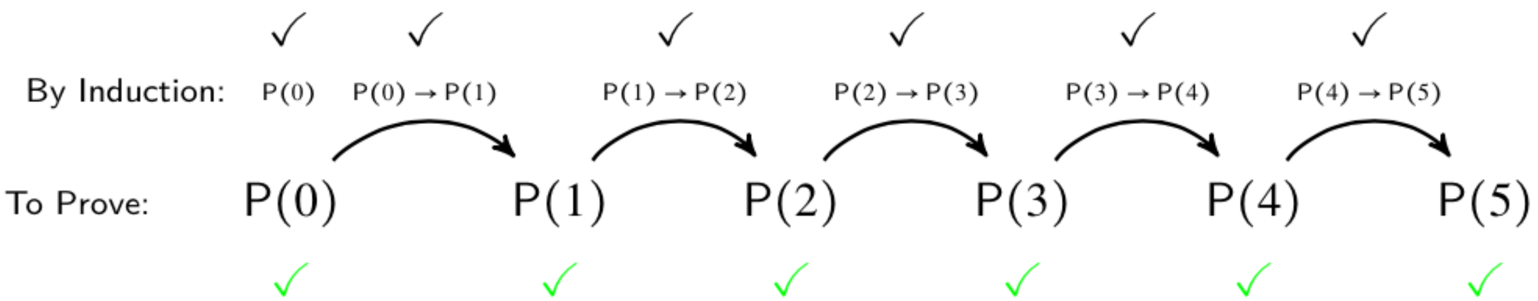
Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall\, k\, (P(k) \rightarrow P(k+1))$$

$$\therefore\ \forall\, n\, P(n)$$

How does this technique prove P(5)?



First, we prove P(0).

Since P(n) → P(n+1) for all n, we have P(0) → P(1).

Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.

Since P(n) → P(n+1) for all n, we have P(1) → P(2).

Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \rightarrow P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. P(k) → P(k+1) Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) Intro ∀ from 2-6
8. ∀ n P(n) Induction Rule 1&7

What can we say about $1 + 2 + 4 + 8 + \ldots + 2^n$

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

- Can we describe the pattern?
  - $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Proving $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

- We could try proving it normally…
  - We want to show that $1 + 2 + 4 + \ldots + 2^n = 2^{n+1}$.
  - So, what do we do now? We can sort of explain the pattern, but that's not a proof…
- We could prove it for n=1, n=2, n=3, … (individually), but that would literally take forever…

Instead, Let's Use Induction

$$P(0)$$
$$\forall\ k\ (P(k)\ \rightarrow\ P(k+1))$$

$$\therefore\ \forall\ n\ P(n)$$

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0
   3. Assume that P(k) is true

Inductive Hypothesis

4. ...
5. Prove P(k+1) is true

Inductive Step

6. P(k) → P(k+1)	Direct Proof Rule
7. ∀ k (P(k) → P(k+1))	Intro ∀ from 2-6
8. ∀ n P(n)	Induction Rule 1&7

Conclusion

5 Steps To Inductive Proofs In English

Proof:

1. "We will show that P(n) is true for every n ≥ 0 by Induction."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !!)
5. "Conclusion: Result follows by induction"

Proving $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$.

Let $P(n)$ be "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". We will show $P(n)$ is true for all natural numbers by induction.

Base Case (n=0):

$$2^0 = 1 = 2 - 1 = 2^{0+1} - 1$$

Induction Hypothesis:

Suppose that $P(k)$ is true for some arbitrary $k \geq 0$.

Induction Step:

WTS: Show $P(k+1)$ (i.e. show $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$)

$1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ (by IH!)

Adding 2^{k+1} to both sides, we get:

$1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly $P(k+1)$.

Thus $P(k)$ is true for all $k \in \mathbb{N}$, by induction.

Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all n ≥ 1.

Let $P(n)$ be "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$"

We go by induction on n.

Base Case:

When n=1, we have 1 = 1(2)/2. So, P(0) is true.

Induction Hypothesis:

Suppose P(k) is true for some k ≥ 1.

Induction Step:

We know $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, by the IH. We want to prove $\sum_{i=1}^{k} i = \frac{(k+1)(k+2)}{2}$.

Note that $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) \underset{\text{IH}}{\overset{\text{By}}{=}} \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

And $\sum_{i=1}^{k} i = \frac{(k+1)(k+2)}{2}$ is exactly P(k+1).

The claim follows for all n ≥ 1, by induction.

Another Pattern

- $20 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $22 - 1 = 4 \ \ - 1 = 3 = 3 \cdot 1$
- $24 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $26 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $28 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- …

Another Example

We want to prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

Let P(n) be "$2^{2n} - 1 = 3k$ for some integer k" for all $n \geq 0$.

We go by induction.

Base Case: When $n = 0$, note that $2^0 - 1 = 0 = 3k$. So, P(0) is true.

Induction Hypothesis: Suppose P(k) is true for some $k \geq 0$.

Induction Step: Note that $2^{2(k+1)} - 1 = (2^{2k})(2^2) - 1$. By the IH, $2^{2k} - 1 = 3j$ for some j. So, $2^{2(k+1)} - 1 = (2^{2k})(2^2) - 1 = (3j + 1)(2^2) - 1 = 12j + 3 = 3(4j + 1)$. It follows that there is some r (namely, $r = 4j + 1$) such that $2^{2(k+1)} - 1 = 3r$.

Since P(0) is true, and $P(k) \rightarrow P(k+1)$ for all $k \geq 0$, the claim is true by induction.