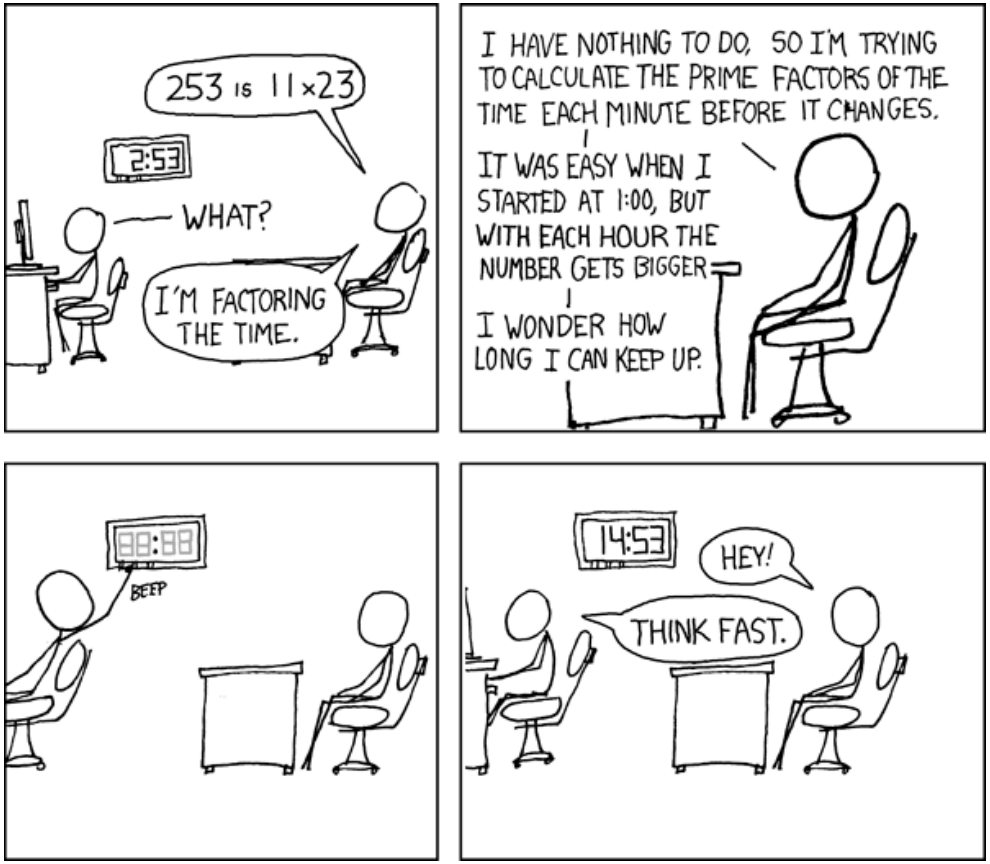
CSE 311: Foundations of Computing

Fall 2014

Lecture 12: Primes, GCD



Basic Applications of mod

- Hashing
- Pseudo random number generation

Hashing

Scenario:

Map a small number of data values from a large domain $\{0, 1, \ldots, M-1\}$ ...

...into a small set of locations $\{0,1,\ldots,n-1\}$ so one can quickly check if some value is present

- $\text{hash}(x) = x \bmod p$ for p a prime close to n
  - or $\text{hash}(x) = (ax+b) \bmod p$
- Depends on all of the bits of the data
  - helps avoid collisions due to similar values
  - need to manage them if they occur

Pseudo-Random Number Generation

Linear Congruential method

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random x_0, a, c, m and produce a long sequence of x_n's

Modular Exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

Exponentiation

- **Compute** 78365^{81453}

- **Compute** $78365^{81453} \bmod 104729$

- **Output is small**
  - need to keep intermediate results small

Repeated Squaring – small and fast

Since $a \bmod m \equiv a \pmod{m}$ for any a

we have $a^2 \bmod m = (a \bmod m)^2 \bmod m$

and $a^4 \bmod m = (a^2 \bmod m)^2 \bmod m$

and $a^8 \bmod m = (a^4 \bmod m)^2 \bmod m$

and $a^{16} \bmod m = (a^8 \bmod m)^2 \bmod m$

and $a^{32} \bmod m = (a^{16} \bmod m)^2 \bmod m$

Can compute $a^k \bmod m$ for $k=2^i$ in only i steps

Fast Exponentiation

```
public static long FastModExp(long base, long exponent, long modulus) {
       long result = 1;
       base = base % modulus;

       while (exponent > 0) {
           if ((exponent % 2) == 1) {
               result = (result * base) % modulus;
                exponent -= 1;
           }
           /* Note that exponent is definitely divisible by 2 here. */
           exponent /= 2;
           base = (base * base) % modulus;
           /* The last iteration of the loop will always be exponent = 1 */
           /* so, result will always be correct. */
       }
       return result;
   }
```

$b^e \bmod m = (b^2)^{e/2} \bmod m$, when e is even)

$b^e \bmod m = (b*(b^{e-1} \bmod m) \bmod m)) \bmod m$

Program Trace

Let M = 104729

$78365^{81453} \bmod M$

$= ((78365 \bmod M) * (78365^{81452} \bmod M)) \bmod M$

$= (78365 * ((78365^2 \bmod M)^{81452/2} \bmod M)) \bmod M$

$= (78365 * ((78852)^{40726} \bmod M)) \bmod M$

$= (78365 * ((78852^2 \bmod M)^{20363} \bmod M)) \bmod M$

$= (78365 * (86632^{20363} \bmod M)) \bmod M$

$= (78365 * ((86632 \bmod M) * (86632^{20362} \bmod M)) \bmod M$

$= \ldots$

$= 45235$

Fast Exponentiation Algorithm

Another way:

$$81453 = 2^{16} + 2^{13} + 2^{12} + 2^{11} + 2^{10} + 2^{9} + 2^{5} + 2^{3} + 2^{2} + 2^{0}$$

$$a^{81453} = a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{10}} \cdot a^{2^{9}} \cdot a^{2^{5}} \cdot a^{2^{3}} \cdot a^{2^{2}} \cdot a^{2^{0}}$$

$$\begin{aligned}
&a^{81453} \bmod m = \\
&(\ldots(((((a^{2^{16}} \bmod m \cdot \\
&\quad a^{2^{13}} \bmod m\ ) \bmod m \cdot \\
&\quad a^{2^{12}} \bmod m) \bmod m \cdot \\
&\quad a^{2^{11}} \bmod m) \bmod m \cdot \\
&\quad a^{2^{10}} \bmod m) \bmod m \cdot \\
&\quad a^{2^{9}} \bmod m) \bmod m \cdot \\
&\quad a^{2^{5}} \bmod m) \bmod m \cdot \\
&\quad a^{2^{3}} \bmod m) \bmod m \cdot \\
&\quad a^{2^{2}} \bmod m) \bmod m \cdot \\
&\quad a^{2^{0}} \bmod m)\ \bmod m
\end{aligned}$$

The fast exponentiation algorithm computes

$a \uparrow n \bmod m$ **using** $O(\log n)$ **multiplications** $\bmod m$

Primality

An integer p greater than 1 is called *prime* if the only positive factors of p are 1 and p.

A positive integer that is greater than 1 and is not prime is called *composite*.

Fundamental Theorem of Arithmetic

Every positive integer greater than 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

Factorization

If n is composite, it has a (non-trivial) factor, f, where $f \leq \sqrt{n}$.

Let n be an arbitrary composite number. Suppose, for contradiction, that all of the factors of n are greater than sqrt(n). Then, since n is composite, there are two factors, a and b, such that $1 < a, b < n$ such that $n = ab$.

Note that a,b > sqrt(n) by assumption.
So, n = ab > sqrt(n)sqrt(n) = n, which is a contradiction. It follows that the original claim is true.

Euclid's Theorem

There are an infinite number of primes.

Proof by contradiction:

Suppose for contradiction that there are n primes for some natural number n. Call them $p_1 < p_2 < \ldots < p_n$. Consider $P = p_1 p_2 \ldots p_n$, and define $Q = P + 1$.

Case 1 (Q is prime). Then, we're done, because Q is larger than any of the primes; so, it is a new prime.

Case 2 (Q is composite). Then, there must be some prime $p \mid q$. Note that since P divides every possible prime, $p \mid P$ as well. It follows that $p \mid (Q - P) \rightarrow p \mid ((P + 1) - P) \rightarrow p \mid 1$. This is impossible, because p must be at least two.

Since both cases lead to a contradiction, the original claim is true.

Famous Algorithmic Problems

- Primality Testing
  - Given an integer n, determine if n is prime
- Factoring
  - Given an integer n, determine the prime factorization of n

Factoring

Factor the following 232 digit number [RSA768]:

123018668453011775513049495838496272077
285356959533479219732245215172640050726
365751874520219978646938995647494277406
384592519255732630345373154826850791702
612214291346167042921431160222124047927
4737794080665351419597459856902143413

1230186684530117755130494958384962720772853569595334792197322452151726400507263657518745202199786469389956474942774063845925192557326303453731548268507917026122142913461670429214311602221240479274737794080665351419597459856902143413

=

33478071698956898786044169848212690817704794983713768568912431388982883793878002287614711652531743087737814467999489

×

36746043666799590428244633799627952632279158164343087642676032283815739666511279233373417143396810270092798736308917

Greatest Common Divisor

GCD(a, b):

Largest integer d such that $d | a$ and $d | b$

- GCD(100, 125) =
- GCD(17, 49) =
- GCD(11, 66) =
- GCD(13, 0) =
- GCD(180, 252) =

GCD and Factoring

$a = 2^3 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 = 46{,}200$

$b = 2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 13 = 204{,}750$

$$\mathrm{GCD}(a, b) = 2^{\min(3,1)} \cdot 3^{\min(1,2)} \cdot 5^{\min(2,3)} \cdot 7^{\min(1,1)} \cdot 11^{\min(1,0)} \cdot 13^{\min(0,1)}$$

Factoring is expensive!

Can we compute GCD(a,b) without factoring?

Useful GCD Fact

If a and b are positive integers, then

$$\gcd(a,b) = \gcd(b, a \bmod b)$$

Proof:

Consider an arbitrary divisor, d, such that d|a and d|b. Note that a = (a div b)b + (a mod b). By definition of d|a, we have (a div b)b + (a mod b) = kd. Since d | b, we also have b = jd. Re-arranging, we see

$$(a \bmod b) = kd – (a \text{ div } b)b = d(k – (a \text{ div } b)j).$$

So, d | (a mod b).

Now, consider an arbitrary divisor, d, such that d | b and d | (a mod b). It follows that (a mod b) = kd. Adding (a div b)b to both sides gives

a = (a mod b) + (a div b)b = kd + (a div b)b = kd + (a div b)jd = d(k + (a div b)j).

So, d | a.

Since all the divisors of a and b are the same as the divisors of b and a mod b, it follows that the greatest divisor of each pair is the same as well.

Euclid's Algorithm

Repeatedly use the GCD fact to reduce numbers until you get $\mathrm{GCD}(x,0)=x$.

gcd(660,126) = gcd(126, 660 mod 126) = gcd(126, 30)
= gcd(30, 126 mod 30) = gcd(30, 6)
= gcd(6, 30 mod 6) = gcd(6, 0)
= 6

Euclid's Algorithm

GCD(x, y) = GCD(y, x mod y)

```
int GCD(int a, int b){ /* a >= b, b > 0 */
    int tmp;
    while (y > 0) {
        tmp = a % b;
        a = b;
        b = tmp;
    }
    return a;
}
```

Example: GCD(660, 126)