CSE 311

Foundations of Computing I

Fall 2014

Review: Division Theorem

Let a be an integer and d a positive integer. Then there are *unique* integers q and r, with $0 \leq r < d$, such that $a = dq + r$.

$q = a$ **div** d $\qquad$ $r = a$ **mod** d

Review: Modular Arithmetic

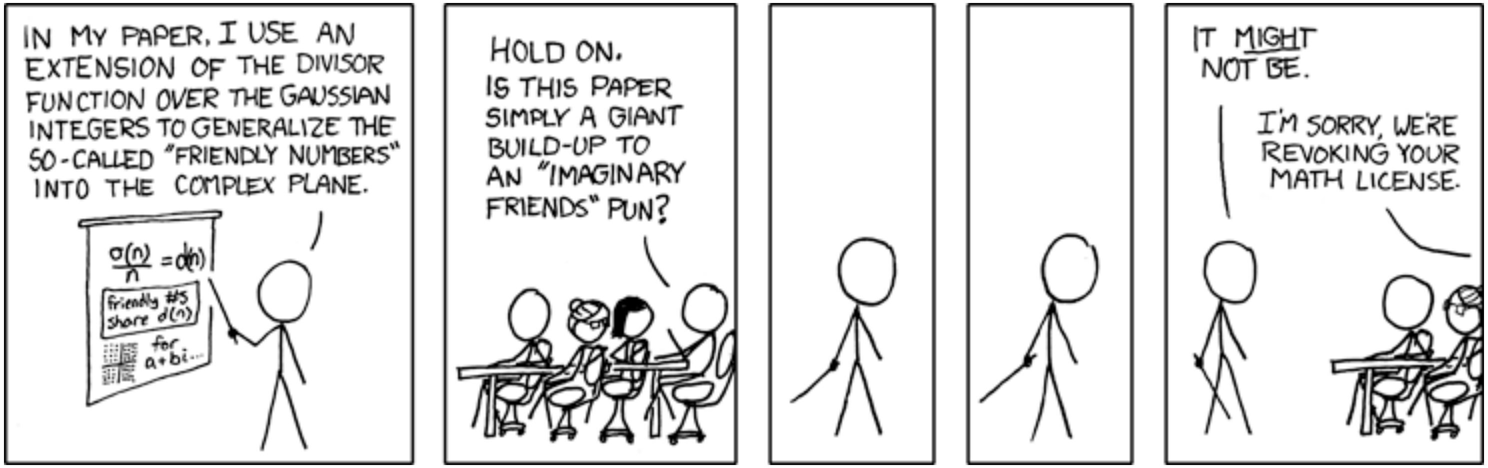
Let a and b be integers, and m be a positive integer. We say a *is congruent to b modulo m* if m divides a – b. We use the notation a ≡ b (mod m) to indicate that a is congruent to b modulo m.

Review: Divisibility

Integers a, b, with $a \neq 0$, we say that a *divides* b if there is an integer k such that $b = ka$. The notation $a \mid b$ denotes “a divides b.”

CSE 311: Foundations of Computing

Fall 2014
Lecture 11: Modular Arithmetic and Applications



Modular Arithmetic: A Property

Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

Proof: Suppose that $a \equiv b \pmod{m}$.

By definition: $a \equiv b \pmod{m}$ implies $m \mid (a - b)$ which by definition implies that $a - b = km$ for some integer k.

Therefore $a=b+km$. Taking both sides modulo m we get

$$a \bmod m=(b+km) \bmod m = b \bmod m.$$

Suppose that $a \bmod m = b \bmod m$.

By the division theorem, $a = mq + (a \bmod m)$ and

$b = ms + (b \bmod m)$ for some integers q,s.

$$\begin{aligned} a - b &= (mq + (a \bmod m)) - (ms + (b \bmod m)) \\ &= m(q - s) + (a \bmod m - b \bmod m) \\ &= m(q - s) \text{ since } a \bmod m = b \bmod m \end{aligned}$$

Therefore $m \mid (a-b)$ and so $a \equiv b \pmod{m}$.

Modular Arithmetic: Another Property

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{a + c \equiv b + d \pmod{m}}$

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. Unrolling definitions gives us some k such that
$a - b = km$, and some j such that $c - d = jm$.

Adding the equations together gives us
$(a + c) - (b + d) = m(k + j)$. Now, re-applying the definition of congruence gives us $a + c \equiv b + d \pmod{m}$.

Modular Arithmetic: Another-nother Property

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $\mathbf{ac \equiv bd \pmod{m}}$

Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. Unrolling definitions gives us some k such that
$a - b = km$, and some j such that $c - d = jm$.

Then, $a = km + b$ and $c = jm + d$. Multiplying both together gives us $ac = (km + b)(jm + d) = kjm^2 + kmd + jmb + bd$.

Re-arranging gives us $ac - bd = m(kjm + kd + jb)$. Using the definition of congruence gives us $ac \equiv bd \pmod{m}$.

Example

Let n be an integer.
Prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

Let's start by looking a a small example:

$0^2 = 0 \equiv 0 \pmod 4$
$1^2 = 1 \equiv 1 \pmod 4$
$2^2 = 4 \equiv 0 \pmod 4$
$3^2 = 9 \equiv 1 \pmod 4$
$4^2 = 16 \equiv 0 \pmod 4$

It looks like
$n \equiv 0 \pmod 2 \rightarrow n^2 \equiv 0 \pmod 4$, and
$n \equiv 1 \pmod 2 \rightarrow n^2 \equiv 1 \pmod 4$.

Case 1 (n is even):
Suppose $n \equiv 0 \pmod 2$.
Then, $n = 2k$ for some k.
So, $n^2 = (2k)^2 = 4k^2$. So, by definition of congruence, $n^2 \equiv 0 \pmod 4$.

Case 2 (n is odd):
Suppose $n \equiv 1 \pmod 2$.
Then, $n = 2k + 1$ for some k.
So, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. So, by definition of congruence, $n^2 \equiv 1 \pmod 4$.

n-bit Unsigned Integer Representation

- Represent integer x as sum of powers of 2:

  If $x = \sum_{i=0}^{n-1} b_i 2^i$ where each $b_i \in \{0,1\}$

  then representation is $b_{n-1}...b_2\ b_1\ b_0$

99 = 64 + 32 + 2 + 1

18 = 16 + 2

- For n = 8:

  99: 0110 0011

  18: 0001 0010

Sign-Magnitude Integer Representation

n-bit signed integers

Suppose $-2^{n-1} < x < 2^{n-1}$

First bit as the sign, n-1 bits for the value

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

99: 0110 0011

-18: 1001 0010

Any problems with this representation?

Two's Complement Representation

n bit signed integers, first bit will still be the sign bit

Suppose $0 \leq x < 2^{n-1}$,

x is represented by the binary representation of x

Suppose $0 \leq x \leq 2^{n-1}$,

$-x$ is represented by the binary representation of $2^n - x$

99 = 64 + 32 + 2 + 1
18 = 16 + 2

For n = 8:

99: 0110 0011
-18: 1110 1110

Sign-Magnitude vs. Two's Complement

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1111	1110	1101	1100	1011	1010	1001	0000	0001	0010	0011	0100	0101	0110	0111

Sign-bit

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111

Two's complement

Two's Complement Representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $2^n - x$

- To compute this: Flip the bits of x then add 1:
  - All 1's string is $2^n - 1$, so

    Flip the bits of x ≡ replace x by $2^n - 1 - x$

Basic Applications of mod

- Hashing
- Pseudo random number generation
- Simple cipher

Hashing

Scenario:

Map a small number of data values from a large domain $\{0, 1, ..., M-1\}$ ...

...into a small set of locations $\{0,1,...,n-1\}$ so one can quickly check if some value is present

- $\mathrm{hash}(x) = x \bmod p$ for p a prime close to n
  - or $\mathrm{hash}(x) = (ax+b) \bmod p$
- Depends on all of the bits of the data
  - helps avoid collisions due to similar values
  - need to manage them if they occur

Pseudo-Random Number Generation

Linear Congruential method

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random x_0, a, c, m and produce a long sequence of x_n's

Simple Ciphers

- **Caesar cipher**, A = 1, B = 2, . . .
  - HELLO WORLD
- **Shift cipher**
  - $f(p) = (p + k) \bmod 26$
  - $f^{-1}(p) = (p - k) \bmod 26$
- **More general**
  - $f(p) = (ap + b) \bmod 26$