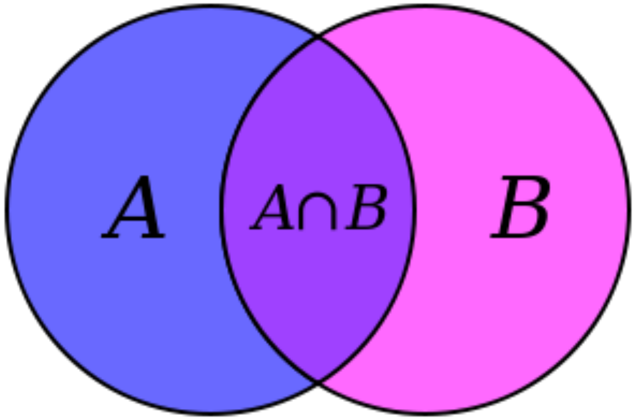
CSE 311



Foundations of Computing I

Fall 2014

Notes on Feedback

I will go over many more notes on Wednesday…but I want to cover two today:

A consistent theme: "worried about exams"

The exam will be easier than the HW. It will consist of "QuickCheck-like" questions.

"More practice plx?"

Section handouts are wayyy longer than you can do.

I'm happy to generate more questions at office hours.

Reminder for HW

For Intro ∃…

Your "c" has to be new **(e. g. cannot be used previously in the proof)**

You should say what variables your "c" depends on.

The order you use Elim ∃ and Elim ∀ in DOES matter!

Reminder: ∃x ∀y P(x,y) IS DIFFERENT FROM ∀y∃x P(x,y)

Proof by Contrapositive: One Strategy for implications

If we assume $\neg q$ and derive $\neg p$, then we have proven $\neg q \rightarrow \neg p$, which is the same as $p \rightarrow q$.

1. $\neg q$	Assumption
...	
3. $\neg p$	
4. $\neg q \rightarrow \neg p$	Direct Proof Rule
5. $p \rightarrow q$	Contrapositive

Proof by Contradiction: One way to prove $\neg p$

If we assume p and derive F (a contradiction), then we have proven $\neg p$.

1.	p	Assumption
...		
3.	F	
4.	$p \rightarrow F$	direct Proof rule
5.	$\neg p \vee F$	Law of Implication: 4
6.	$\neg p$	Identity: 5

Even and Odd

Even(x) $\equiv \exists y \ (x=2y)$
Odd(x) $\equiv \exists y \ (x=2y+1)$
Domain: Integers

Prove: "No integer is both even and odd."

English proof: $\neg \exists x\, (\mathrm{Even}(x) \wedge \mathrm{Odd}(x))$

$\equiv \forall x\, \neg(\mathrm{Even}(x) \wedge \mathrm{Odd}(x))$

We go by contradiction. Let x be any integer and suppose that it is both even and odd. Then x=2k for some integer k and x=2m+1 for some integer m. Therefore 2k=2m+1 and hence k=m+½.

But two integers cannot differ by ½ so this is a contradiction. So, no integer is both even and odd.

Rational Numbers

- **A real number x is *rational* iff there exist integers p and q with q≠0 such that x=p/q.**

$\text{Rational}(x) \equiv \exists p\, \exists q\ ((x=p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land q \neq 0)$

- **Prove: If x and y are rational then xy is rational**

$\forall x\, \forall y\ ((\text{Rational}(x) \land \text{Rational}(y)) \rightarrow \text{Rational}(xy))$

Domain: Real numbers

Rationality

Rational(x) $\equiv \exists p\, \exists q\ ((x=p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge q \neq 0)$
Domain: Reals

Prove: "If x and y are rational then xy is rational."

Let x and y be rational numbers. Then, $x = a/b$ for some integers a, b, where $b \neq 0$, and $y = c/d$ for some integers c,d, where $d \neq 0$.

Note that $xy = (ac)/(bd)$.

Since b and d are both non-zero, so is bd; furthermore, ac and bd are integers. It follows that xy is rational, by definition of rational.

Proofs

- Formal proofs follow simple well-defined rules and should be easy to check
  - In the same way that code should be easy to execute

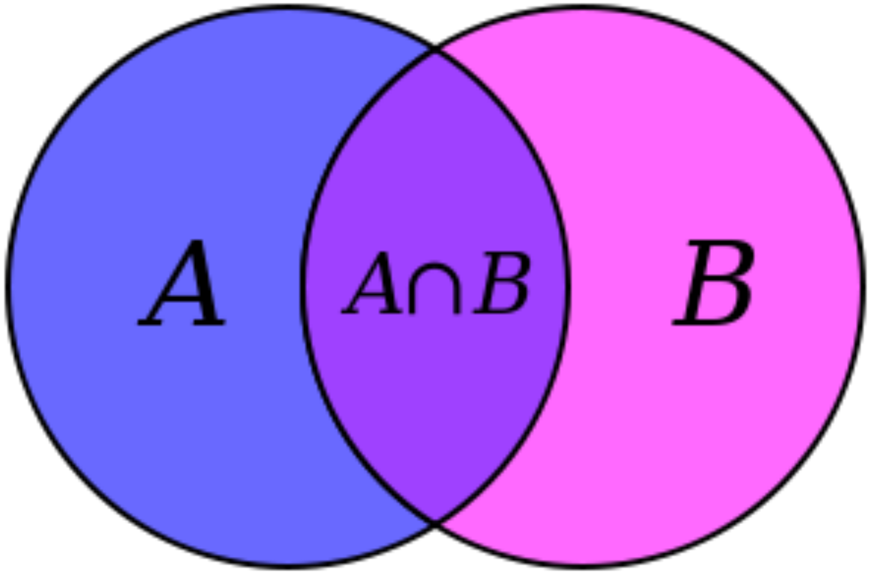
- English proofs correspond to those rules but are designed to be easier for humans to read
  - Easily checkable in principle

- Simple proof strategies already do a lot
  - Later we will cover a specific strategy that applies to loops and recursion (mathematical induction)

CSE 311: Foundations of Computing

Fall 2014

Lecture 9: Set Theory



Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** N = {0, 1, 2, …}
$\mathbb{Z}$ is the set of **Integers**; Z = {…, -2, -1, 0, 1, 2, …}
$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48
$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, π
[n] is the set {1, 2, …, n} when n is a natural number
{} = ∅ is the **empty set**; the *only* set with no elements

EXAMPLES
Are these sets?
A = {1, 1}
B = {1, 3, 2}
C = {□, 1}
D = {{}, 17}
E = {1, 2, 7, cat, dog, ∅, α}

We say 2 $\in$ E; 3 $\notin$ E.

They're all sets.
Note {1} = {1, 1}.

Sets are un-typed. Sets can contain other sets.

Definitions

- A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\, (x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\, (x \in A \rightarrow x \in B)$$

$A = \{1, 2, 3\}$
$B = \{3, 4, 5\}$
$C = \{3, 4\}$

QUESTIONS

$\varnothing \subseteq A$? **Yes. In fact, $\varnothing \subseteq X$ for any set X.**

$A \subseteq B$? **No. $3 \in A$, but that's not true for B.**

$C \subseteq B$? **Yes, $3 \in B$, $4 \in B$.**

Definitions

- A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\,(x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\,(x \in A \rightarrow x \in B)$$

- Note: $(\boldsymbol{A} = \boldsymbol{B}) \equiv (\boldsymbol{A} \subseteq \boldsymbol{B}) \wedge (\boldsymbol{B} \subseteq \boldsymbol{A})$

Building Sets from Predicates

- The following says "S is the set of all x's where P(x) is true.

$$S = \{x : P(x)\}$$

- The following says "those elements of S for which P(x) is true."

$$S = \{x \in A : P(x)\}$$

- "All the real numbers less than one"
  - $\{x \in \mathbb{R} : x < 1\}$
- "All the powers of two that happen to be odd."
  - $\{x \in \mathbb{N} : \exists k\ (x = 2k+1) \wedge \exists j\ (x = 2^j)\}$

Set Operations

$$A \cup B = \{\, x : (x \in A) \vee (x \in B)\}$$ **Union**

$$A \cap B = \{\, x : (x \in A) \wedge (x \in B)\}$$ **Intersection**

$$A \setminus B = \{\, x : (x \in A) \wedge (x \notin B)\}$$ **Set Difference**

A = {1, 2, 3}
B = {4, 5, 6}
C = {3, 4}

<u>QUESTIONS</u>

Using A, B, C and set operations, make…

[6] = $A \cup B = A \cup B \cup C$

{3} = $C \setminus B = A \setminus B = A \cap B$

{1,2} = $A \setminus C = (A \cup B) \setminus C$

More Set Operations

$$A \oplus B = \{\, x : (x \in A) \oplus (x \in B) \}$$

Symmetric Difference

$$\overline{A} = \{\, x : x \notin A \,\}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 4, 2, 6}
C = {1, 2, 3, 4}

<u>QUESTIONS</u>

Let S = {1, 2}.

If the universe is A, then $\overline{S}$ is... **A $\setminus$ S = {3}**

If the universe is B, then $\overline{S}$ is... **B $\setminus$ S = {4, 6}**

If the universe is C, then $\overline{S}$ is... **C $\setminus$ S = {3, 4}**

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) = \{\, B : B \subseteq A \,\}$$

- Let Days = {M, W, F}. Suppose we wanted to know the possible ways that we could allocate class days to be cancelled. Let's call this set $\mathcal{P}(\text{Days})$.

e.g. $\mathcal{P}(\emptyset)$ = {Ø}

e.g.
$$\mathcal{P}(\text{Days}) = \{ \\ \emptyset, \\ \{M\}, \{W\}, \{F\}, \\ \{M, W\}, \{W, F\}, \{M, F\}, \\ \{M, W, F\} \\ \}$$

Cartesian Product

$$A \times B = \{ (a, b) : a \in A, b \in B \}$$

R x R is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

Z x Z is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

A × $\emptyset = \{(a, b) : a \in A \land b \in \emptyset\} = \{(a, b) : a \in A \land F\} = \emptyset$

Russell's Paradox

$$S = \{ x : x \notin x \}$$

Suppose for contradiction that S ∈ S. Then, by definition of S, S ∉ S, but that's a contradiction.

Suppose for contradiction that S ∉ S. Then, by definition of the set comprehension, S ∈ S, but that's a contradiction.

This is reminiscent of the truth value of the statement "This statement is false."