CSE 311

Foundations of Computing I

Fall 2014

Remember, the site is…

http://tinyurl.com/ynlecture

Get started on the green handout!

Negations of Quantifiers

- $\forall x$ PurpleFruit(x)
  - **"All fruits are purple"**
- What is $\neg\forall x$ PurpleFruit(x)?
  - **"Not all fruits are purple"**

- How about $\exists x$ PurpleFruit(x)**?**
  - **"There is a purple fruit"**
  - **If it's the negation, all situations should be covered by a statement and its negation**
  - **Consider the domain** {Orange}**: Neither statement is true!**
  - **No!**

- How about $\exists x\ \neg$PurpleFruit(x)**?**
  - **"There is a fruit that isn't purple"**
  - **Yes!**

Domain:
Fruit

PurpleFruit(x)

De Morgan's Laws for Quantifiers

$$\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$$
$$\neg \exists x\ P(x) \equiv \forall x\ \neg P(x)$$

"There is no largest integer"

$$\begin{aligned}\neg \exists x \forall y\ (x \geq y) &\equiv \forall x \neg \forall y\ (x \geq y)\\ &\equiv \forall x\ \exists y\ \neg (x \geq y)\\ &\equiv \forall x\ \exists y\ (y > x)\end{aligned}$$

"For every integer there is a larger integer"

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists y\ \text{Greater}(y, x)$

$\equiv \exists z\ \text{Greater}(z, x)$

truth value:

doesn't depend on y **or** z **"bound variables"**

does depend on x **"free variable"**

quantifiers only act on free variables of the formula they quantify

$$\forall x\ (\exists y\ (P(x,y) \rightarrow \forall x\ Q(y, x)))$$

scope of quantifiers

$\exists x\ (P(x) \wedge Q(x))$ **vs.** $\exists x\ P(x) \wedge \exists x\ Q(x)$

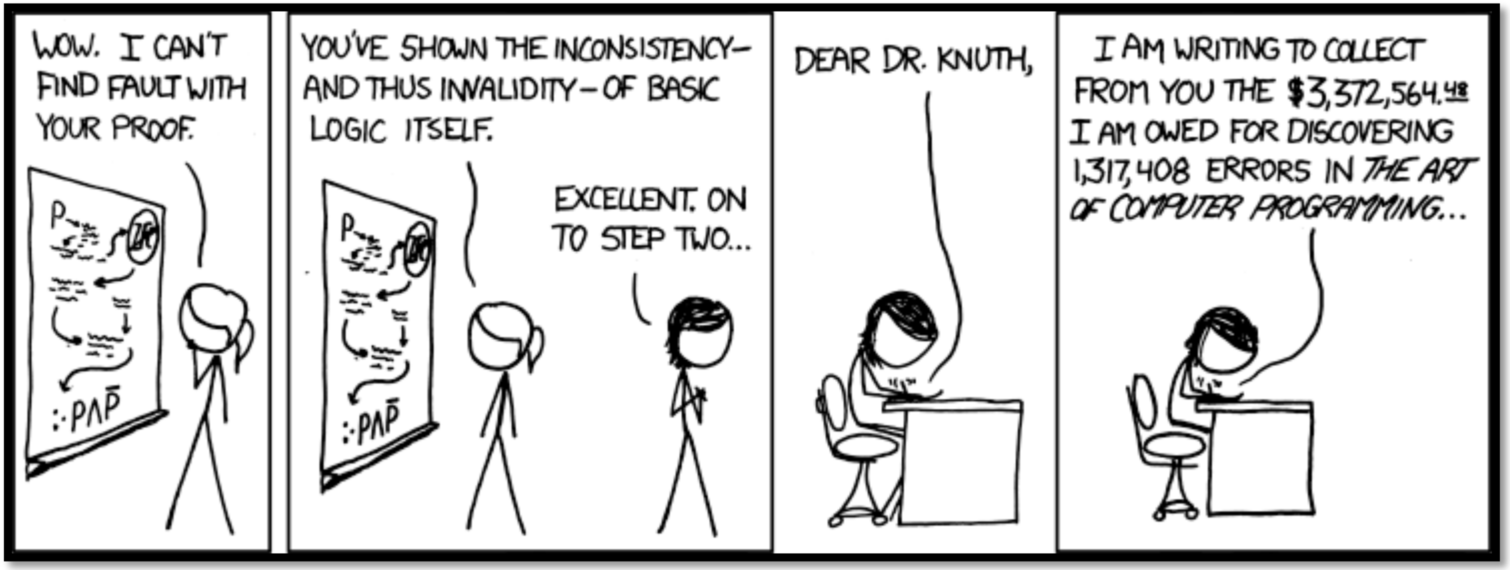
This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

CSE 311: Foundations of Computing

Fall 2014

Lecture 6: Predicate Logic, Logical Inference



Quantifiers are Like Code

$$\forall x\, (\exists y\, (P(x,y) \rightarrow \forall x\, Q(y, x)))$$

```
public boolean blue() {
    for (T x : DOMAIN) {
        if (!green(x)) {
            return false;
        }
    }
    return true;
}
```

```
public boolean red(T z, T y) {
    for (T x : DOMAIN) {
        if (!Q(y,x)) {
            return false;
        }
    }
    return true;
}
```

```
public boolean green(T x) {
    for (T y : DOMAIN) {
        if (!P(x,y) || red(x,y)) {
            return true;
        }
    }
    return false;
}
```

Notice that we renamed x in red, because we define another x inside.

We recommend that you NOT re-use the same variable like this.

Turtles All The Way Down

If the tortoise walks at a rate of one node per step, and the hare walks at a rate of two nodes per step, then the distance between them increases by one node per step.

$$(p \land q) \rightarrow r$$

If the tortoise is on node x, and the hare is on node 2x, then the distance between them increases by one node per step

$$(\forall x\,(\mathrm{OnNode}(\mathrm{Tortoise}, \mathrm{x}) \land \mathrm{OnNode}(\mathrm{Hare}, 2\mathrm{x}))) \rightarrow p$$

OnNode(x, y)

Domain:
Non-negative Integers

Nested Quantifiers

- Bound variable names don't matter

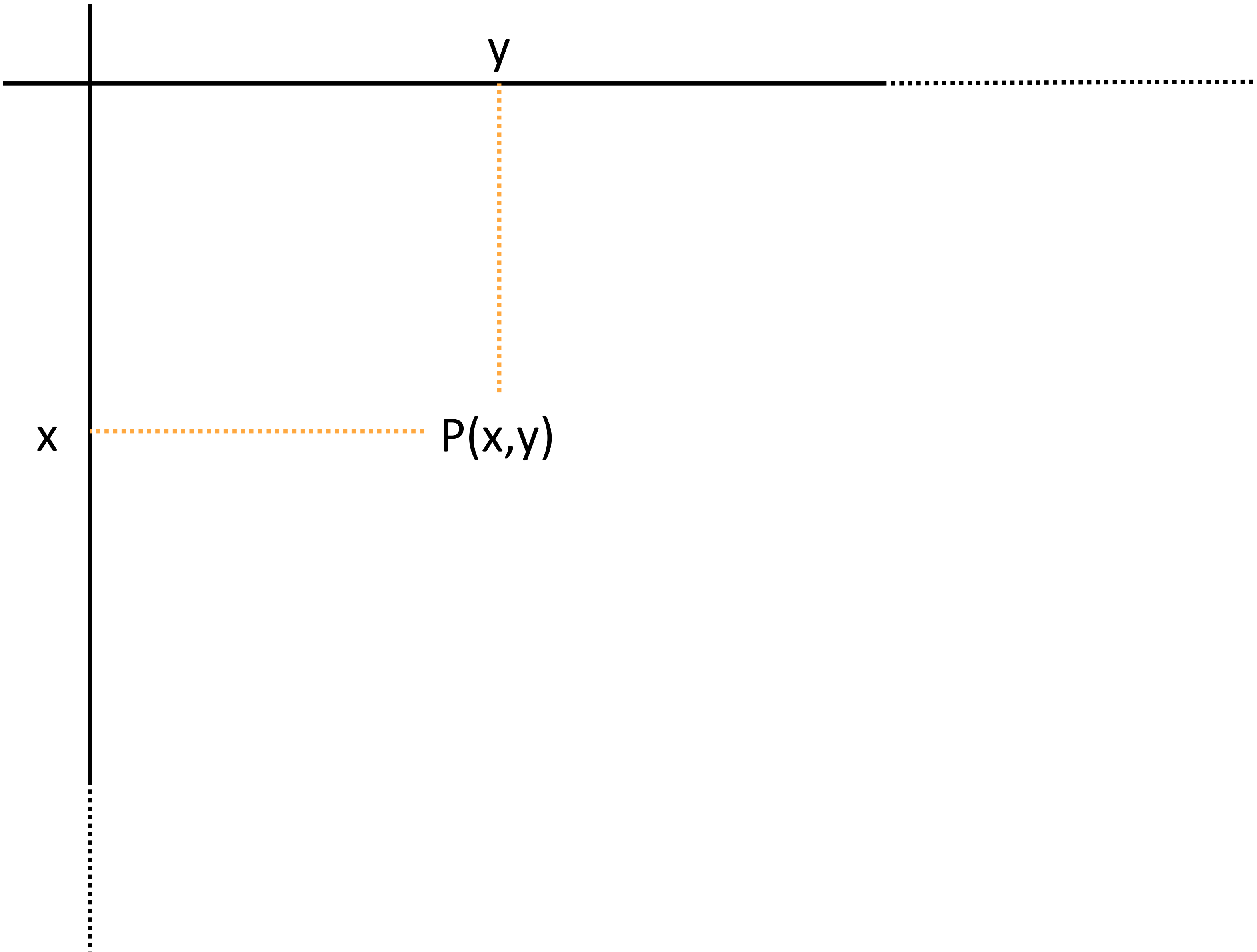
$$\forall x\ \exists y\ P(x, y) \equiv \forall a\ \exists b\ P(a, b)$$

- Positions of quantifiers can sometimes change

$$\forall x\ (Q(x) \land \exists y\ P(x, y)) \equiv \forall x\ \exists y\ (Q(x) \land P(x, y))$$

- But: order is important...

Predicate with Two Variables



Quantification with Two Variables

expression	when **true**	when **false**
$\forall x \forall y P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other
- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If p and p → q are both true then q must be true

- Write this rule as

$$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.

- Therefore, by modus ponens:
  - You have a 311 class today.

Proofs

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	given
2.	$p \rightarrow q$	given
3.	$q \rightarrow r$	given
4.	q	modus ponens from 1 and 2
5.	r	modus ponens from 3 and 4

Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	given
2.	$\neg q$	given
3.	$\neg q \rightarrow \neg p$	contrapositive of 1
4.	$\neg p$	modus ponens from 2 and 3

Inference Rules

- Each inference rule is written as:
  ...which means that if both A and B are true then you can infer C and you can infer D.
  – For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies

$$\frac{A, B}{\therefore C, D}$$

- Sometimes rules don't need anything to start with. These rules are called axioms:
  – e.g. *Excluded Middle Axiom*

$$\therefore p \vee \neg p$$

Simple Propositional Inference Rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

$$\frac{p \land q}{\therefore\ p,\ q}$$

$$\frac{p,\ q}{\therefore\ p \land q}$$

$$\frac{p \lor q\ ,\ \neg p}{\therefore\ q}$$

$$\frac{p}{\therefore\ p \lor q,\ q \lor p}$$

$$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

$$\frac{p \Rightarrow q}{\therefore\ p \rightarrow q}$$

Direct Proof Rule

Not like other rules

Important: Applications of inference rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \rightarrow q$ given

~~2. $(p \vee r) \rightarrow q$ intro $\vee$ from 1.~~

Does not follow! e.g . p=F, q=F, r=T