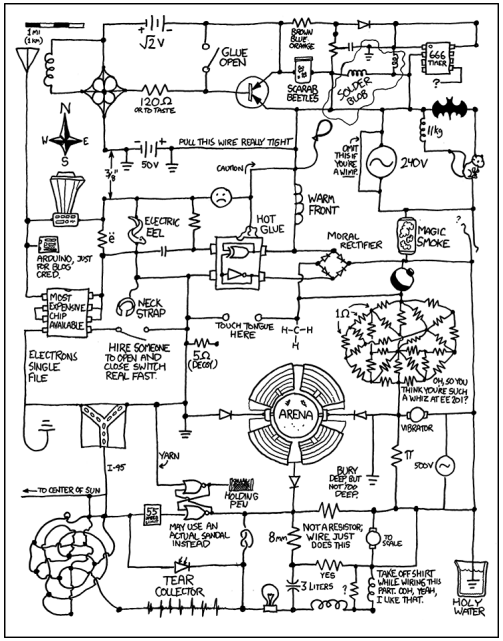
CSE 311



Foundations of Computing I

Fall 2014

Remember, the site is…

http://tinyurl.com/ynlecture

Get started on the blue handout!

Recall…in boolean logic,

A • B is A and B

A + B is A or B

A' is not A

Administrivia

Homework 1 Due Today

- Hand in at start of class

Homework 2 Available Online Later Today

A Combinational Logic Example

Sessions of Class:

We would like to compute the number of lectures or quiz sections remaining *at the start* of a given day of the week.

- **Inputs:** Day of the Week, Lecture/Section flag
- **Output:** Number of sessions left

Examples: Input: (Wednesday, Lecture) Output: **2**

Input: (Monday, Section) Output: **1**

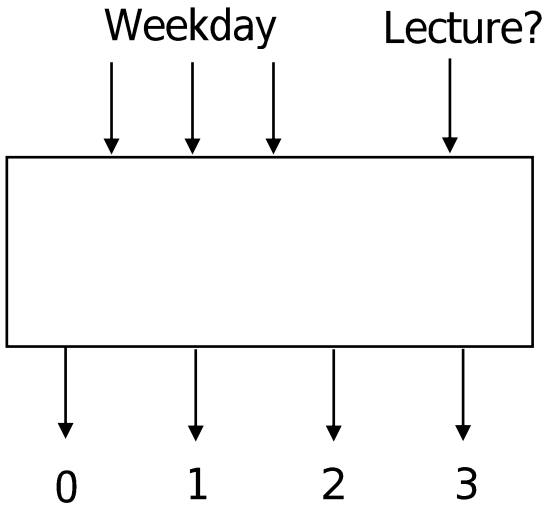
Implementation in Software

```
public int classesLeftInMorning(weekday, lecture_flag) {
    switch (day) {
        case SUNDAY:
        case MONDAY:
            return lecture_flag ? 3 : 1;
        case TUESDAY:
        case WEDNESDAY:
            return lecture_flag ? 2 : 1;
        case THURSDAY:
            return lecture_flag ? 1 : 1;
        case FRIDAY:
            return lecture_flag ? 1 : 0;
        case SATURDAY:
            return lecture_flag ? 0 : 0;
   }
}
```

Implementation with Combinational Logic

Encoding:

- How many bits for each input/output?
- Binary number for weekday
- One bit for each possible output



Defining Our Inputs!

```
public int classesLeftInMorning(weekday, lecture_flag) {
    switch (day) {
        case SUNDAY:
        case MONDAY:
            return lecture_flag ? 3 : 1;
        case TUESDAY:
        case WEDNESDAY:
            return lecture_flag ? 2 : 1;
        case THURSDAY:
            return lecture_flag ? 1 : 1;
        case FRIDAY:
            return lecture_flag ? 1 : 0;
        case SATURDAY:
            return lecture_flag ? 0 : 0;
   }
}
```

Weekday	Number	Binary
Sunday	0	$(000)_2$
Monday	1	$(001)_2$
Tuesday	2	$(010)_2$
Wednesday	3	$(011)_2$
Thursday	4	$(100)_2$
Friday	5	$(101)_2$
Saturday	6	$(110)_2$

Converting to a Truth Table!

Weekday	Number	Binary
Sunday	0	$(000)_2$
Monday	1	$(001)_2$
Tuesday	2	$(010)_2$
Wednesday	3	$(011)_2$
Thursday	4	$(100)_2$
Friday	5	$(101)_2$
Saturday	6	$(110)_2$

Weekday	Lecture?	c0	c1	c2	c3
000	0	0	1	0	0
000	1	0	0	0	1
001	0	0	1	0	0
001	1	0	0	0	1
010	0	0	1	0	0
010	1	0	0	1	0
011	0	0	1	0	0
011	1	0	0	1	0
100	-	0	1	0	0
101	0	1	0	0	0
101	1	0	1	0	0
110	-	1	0	0	0
111	-	-	-	-	-

Truth Table to Logic (Part 1)

c3 = (DAY == SUN and LEC) or (DAY == MON and LEC)

c3 = (d2 == 0 && d1 == 0 && d0 == 0 && L == 1) ||
(d2 == 0 && d1 == 0 && d0 == 1 && L == 1)

c3 = d2'•d1'•d0'•L + d2'•d1'•d0•L

DAY	d2 d1 d0	L	c0	c1	c2	c3
SunS	000	0	0	1	0	0
SunL	000	1	0	0	0	1
MonS	001	0	0	1	0	0
MonL	001	1	0	0	0	1
TueS	010	0	0	1	0	0
TueL	010	1	0	0	1	0
WedS	011	0	0	1	0	0
WedL	011	1	0	0	1	0
Thu	100	-	0	1	0	0
FriS	101	0	1	0	0	0
FriL	101	1	0	1	0	0
Sat	110	-	1	0	0	0
-	111	-	-	-	-	-

Truth Table to Logic (Part 2)

c3 = d2'•d1'•d0'•L + d2'•d1'•d0•L

c2 = (DAY == TUE and LEC) or
(DAY == WED and LEC)

c2 = d2'•d1•d0'•L + d2'•d1•d0•L

DAY	d2 d1 d0	L	c0	c1	c2	c3
SunS	000	0	0	1	0	0
SunL	000	1	0	0	0	1
MonS	001	0	0	1	0	0
MonL	001	1	0	0	0	1
TueS	010	0	0	1	0	0
TueL	010	1	0	0	1	0
WedS	011	0	0	1	0	0
WedL	011	1	0	0	1	0
Thu	100	-	0	1	0	0
FriS	101	0	1	0	0	0
FriL	101	1	0	1	0	0
Sat	110	-	1	0	0	0
-	111	-	-	-	-	-

Truth Table to Logic (Part 3)

c3 = d2'•d1'•d0'•L + d2'•d1'•d0•L

c2 = d2'•d1 •d0'•L + d2'•d1 •d0•L

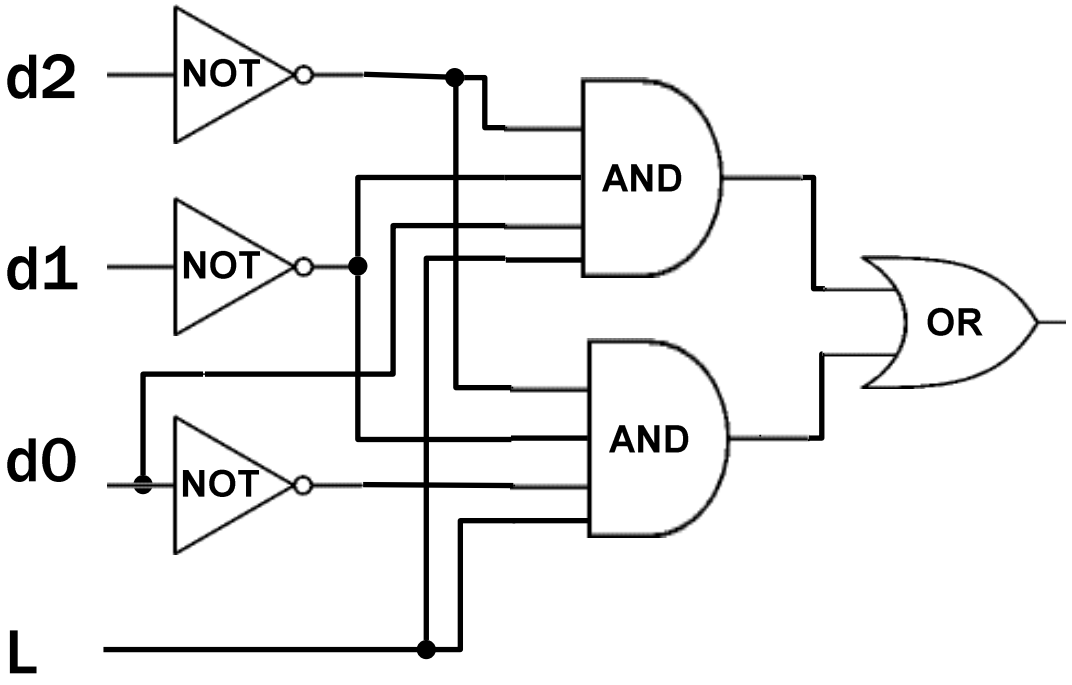
c1 = On your homework for next week!

c0 = d2•d1'• d0 •L' + d2•d1 •d0'

DAY	d2 d1 d0	L	c0	c1	c2	c3
SunS	000	0	0	1	0	0
SunL	000	1	0	0	0	1
MonS	001	0	0	1	0	0
MonL	001	1	0	0	0	1
TueS	010	0	0	1	0	0
TueL	010	1	0	0	1	0
WedS	011	0	0	1	0	0
WedL	011	1	0	0	1	0
Thu	100	-	0	1	0	0
FriS	101	0	1	0	0	0
FriL	101	1	0	1	0	0
Sat	110	-	1	0	0	0
-	111	-	-	-	-	-

Logic to Gates

$c3 = d2' \cdot d1' \cdot d0' \cdot L + d2' \cdot d1' \cdot d0 \cdot L$



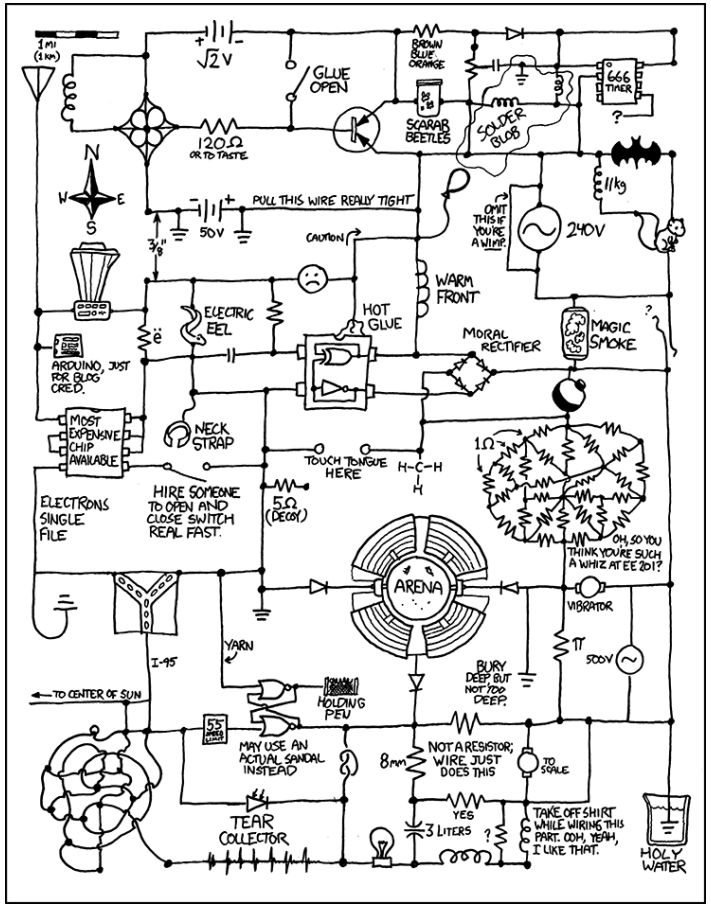
Multi-input AND gates

DAY	d2	d1	d0	L	c0	c1	c2	c3
SunS	0	0	0	0	0	1	0	0
SunL	0	0	0	1	0	0	0	1
MonS	0	0	1	0	0	1	0	0
MonL	0	0	1	1	0	0	0	1
TueS	0	1	0	0	0	1	0	0
TueL	0	1	0	1	0	0	1	0
WedS	0	1	1	0	0	1	0	0
WedL	0	1	1	1	0	0	1	0
Thu	1	0	0	-	0	1	0	0
FriS	1	0	1	0	1	0	0	0
FriL	1	0	1	1	0	1	0	0
Sat	1	1	0	-	1	0	0	0
-	1	1	1	-	-	-	-	-

CSE 311: Foundations of Computing

Fall 2014

Lecture 4: Boolean Algebra and Circuits



Boolean Algebra

- **Boolean algebra to circuit design**
- **Boolean algebra**
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { ’ }
  - such that the following axioms hold:

1. the set B contains at least two elements: 0, 1

For any a, b, c in B:

2. closure:	a + b is in B	a • b is in B
3. commutativity:	a + b = b + a	a • b = b • a
4. associativity:	a + (b + c) = (a + b) + c	a • (b • c) = (a • b) • c
5. identity:	a + 0 = a	a • 1 = a
6. distributivity:	a + (b • c) = (a + b) • (a + c)	a • (b + c) = (a • b) + (a • c)
7. complementarity:	a + a’ = 1	a • a’ = 0

Axioms and Theorems of Boolean Algebra

identity:		
	1. $X + 0 = X$	1D. $X \cdot 1 = X$
null:		
	2. $X + 1 = 1$	2D. $X \cdot 0 = 0$
idempotency:		
	3. $X + X = X$	3D. $X \cdot X = X$
involution:		
	4. $(X')' = X$	
complementarity:		
	5. $X + X' = 1$	5D. $X \cdot X' = 0$
commutativity:		
	6. $X + Y = Y + X$	6D. $X \cdot Y = Y \cdot X$
associativity:		
	7. $(X + Y) + Z = X + (Y + Z)$	7D. $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$
distributivity:		
	8. $X \cdot (Y + Z) = (X \cdot Y) + (X \cdot Z)$	8D. $X + (Y \cdot Z) = (X + Y) \cdot (X + Z)$

Axioms and Theorems of Boolean Algebra

uniting:

9. $X \cdot Y + X \cdot Y' = X$

9D. $(X + Y) \cdot (X + Y') = X$

absorption:

10. $X + X \cdot Y = X$

10D. $X \cdot (X + Y) = X$

11. $(X + Y') \cdot Y = X \cdot Y$

11D. $(X \cdot Y') + Y = X + Y$

factoring:

12. $(X + Y) \cdot (X' + Z) = X \cdot Z + X' \cdot Y$

12D. $X \cdot Y + X' \cdot Z = (X + Z) \cdot (X' + Y)$

consensus:

13. $(X \cdot Y) + (Y \cdot Z) + (X' \cdot Z) = X \cdot Y + X' \cdot Z$

13D. $(X + Y) \cdot (Y + Z) \cdot (X' + Z) = (X + Y) \cdot (X' + Z)$

de Morgan's:

14. $(X + Y + \ldots)' = X' \cdot Y' \cdot \ldots$

14D. $(X \cdot Y \cdot \ldots)' = X' + Y' + \ldots$

Proving Theorems (Rewriting)

Using the laws of Boolean Algebra:

prove the theorem: $X \bullet Y + X \bullet Y' = X$

distributivity (8)	$X \bullet Y + X \bullet Y' = X \bullet (Y + Y')$
complementarity (5)	$= X \bullet (1)$
identity (1D)	$= X$

prove the theorem: $X + X \bullet Y = X$

identity (1D)	$X + X \bullet Y = X \bullet 1 + X \bullet Y$
distributivity (8)	$= X \bullet (1 + Y)$
uniting (2)	$= X \bullet (1)$
identity (1D)	$= X$

Proving Theorems (Truth Table)

Using complete truth table:

For example, de Morgan's Law:

$(X + Y)' = X' \bullet Y'$

NOR is equivalent to AND with inputs complemented

X	Y	X'	Y'	(X + Y)'	X' • Y'
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

$(X \bullet Y)' = X' + Y'$

NAND is equivalent to OR with inputs complemented

X	Y	X'	Y'	(X • Y)'	X' + Y'
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

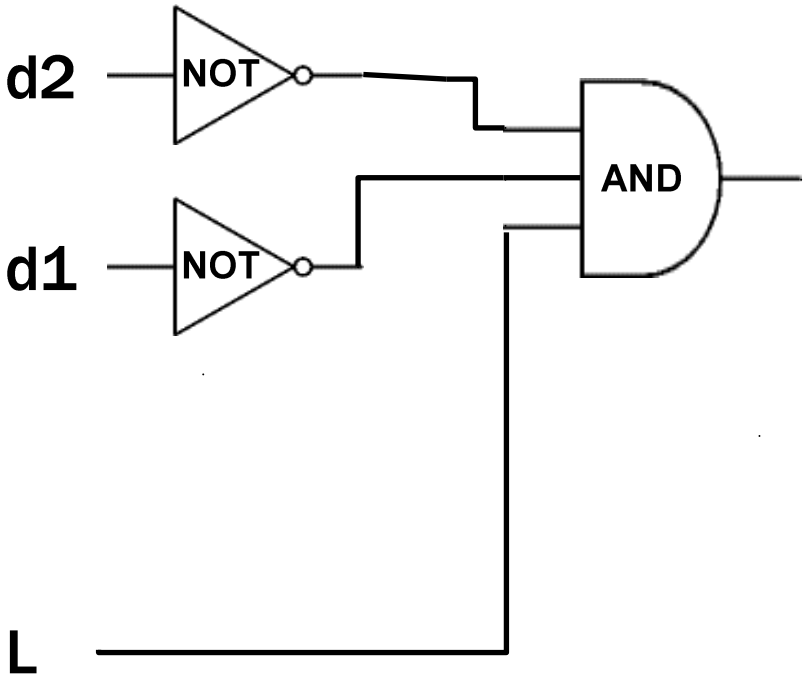
Simplifying using Boolean algebra

c3 = d2'•d1'•d0'•L + d2'•d1'•d0•L

= d2'•d1'•(d0' + d0)•L

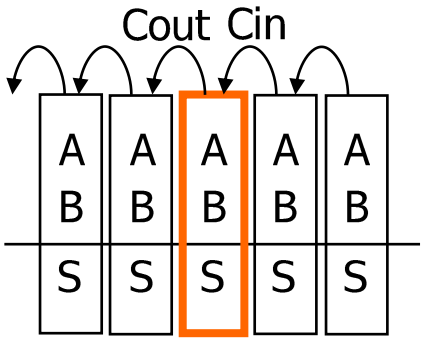
= d2'•d1'•(1)•L

= d2'•d1'•L



1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	Cin	Cout	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$S = A' B' Cin + A' B Cin' + A B' Cin' + A B Cin$$

$$Cout = A' B Cin + A B' Cin + A B Cin' + A B Cin$$

Apply Theorems to Simplify Expressions

The theorems of Boolean algebra can simplify expressions

- **e.g., full adder's carry-out function**

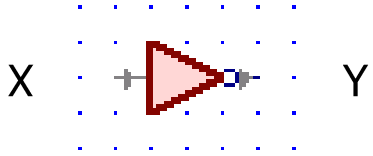
Cout = A' B Cin + A B' Cin + A B Cin' + A B Cin
= A' B Cin + A B' Cin + A B Cin' + A B Cin + A B Cin
= A' B Cin + A B Cin + A B' Cin + A B Cin' + A B Cin
= (A' + A) B Cin + A B' Cin + A B Cin' + A B Cin
= (1) B Cin + A B' Cin + A B Cin' + A B Cin
= B Cin + A B' Cin + A B Cin' + A B Cin + A B Cin
= B Cin + A B' Cin + A B Cin + A B Cin' + A B Cin
= B Cin + A (B' + B) Cin + A B Cin' + A B Cin
= B Cin + A (1) Cin + A B Cin' + A B Cin
= B Cin + A Cin + A B (Cin' + Cin)
= B Cin + A Cin + A B (1)
= B Cin + A Cin + A B

adding extra terms creates new factoring opportunities

Gates Again!

NOT

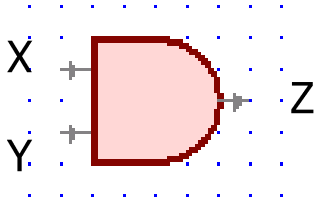
X' $\bar{X}$ $\neg X$



X	Y
0	1
1	0

AND

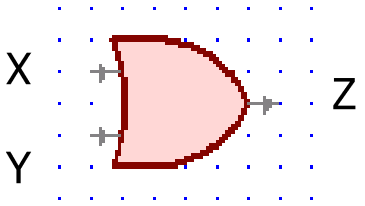
$X \cdot Y$ XY $X \wedge Y$



X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

OR

$X + Y$ $X \vee Y$

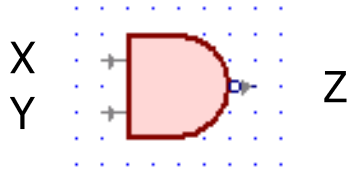


X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

More Gates!

NAND

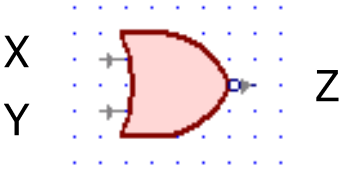
$\neg(X \wedge Y)$ $(XY)'$



X	Y	Z
0	0	1
0	1	1
1	0	1
1	1	0

NOR

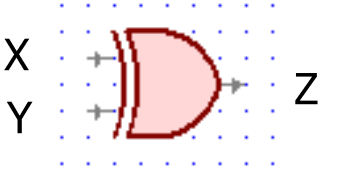
$\neg(X \vee Y)$ $(X + Y)'$



X	Y	Z
0	0	1
0	1	0
1	0	0
1	1	0

XOR

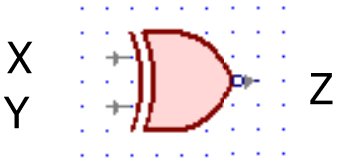
$X \oplus Y$



X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	0

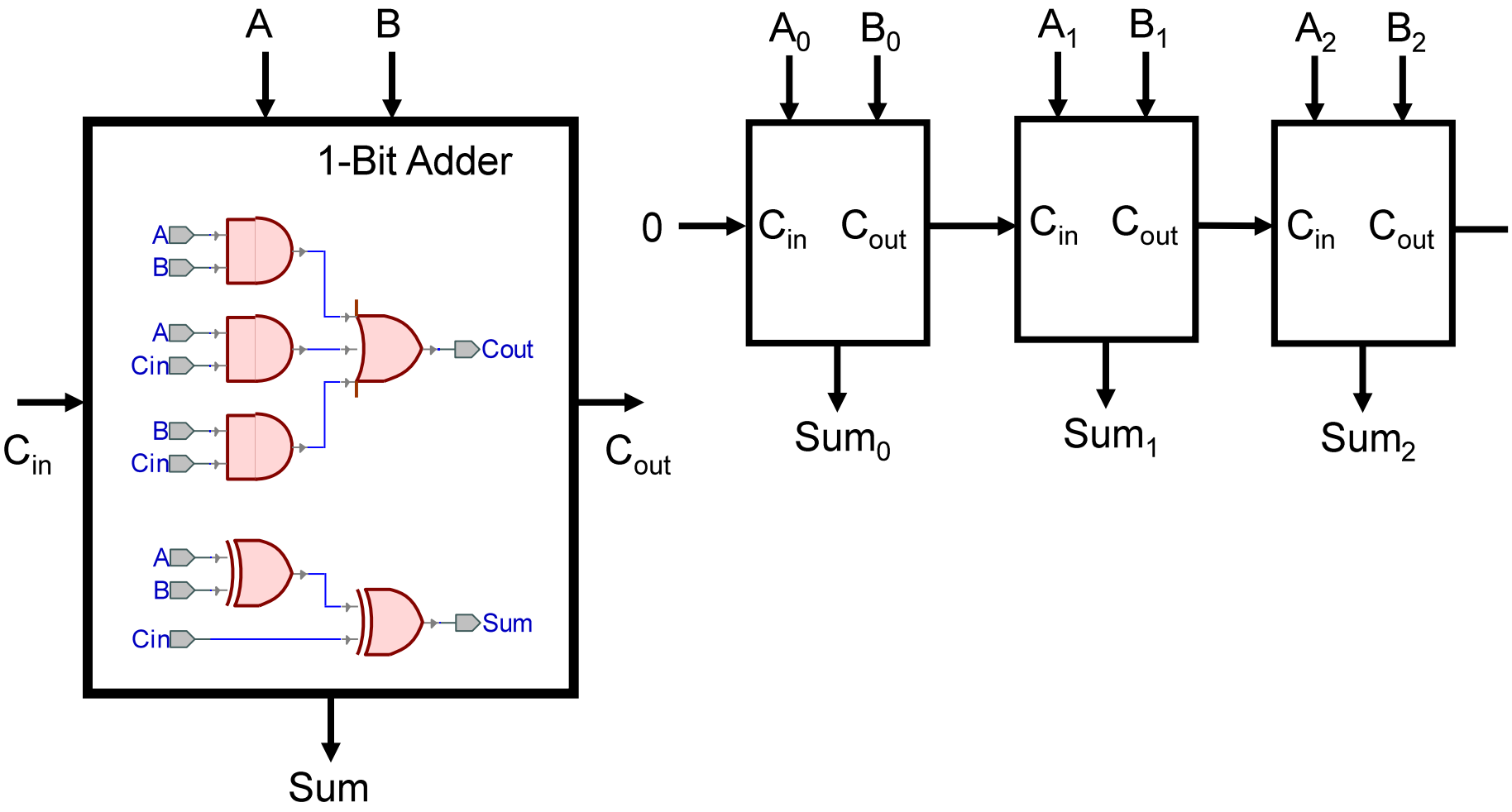
XNOR

$X \leftrightarrow Y$



X	Y	Z
0	0	1
0	1	0
1	0	0
1	1	1

A 2-bit Ripple-Carry Adder



Mapping Truth Tables to Logic Gates

Given a truth table:

1. Write the Boolean expression
2. Minimize the Boolean expression
3. Draw as gates
4. Map to available gates

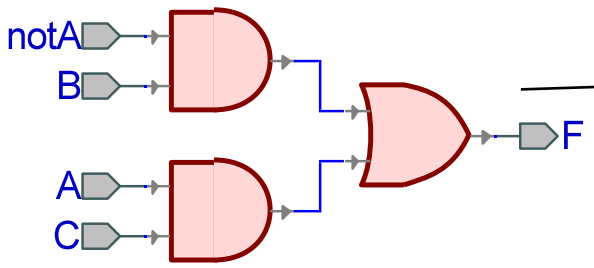
A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

(1)

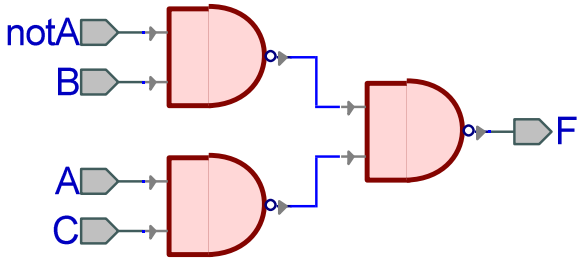
(2)

F = A'BC'+A'BC+AB'C+ABC
= A'B(C'+C)+AC(B'+B)
= A'B+AC

(3)



(4)



Canonical Forms

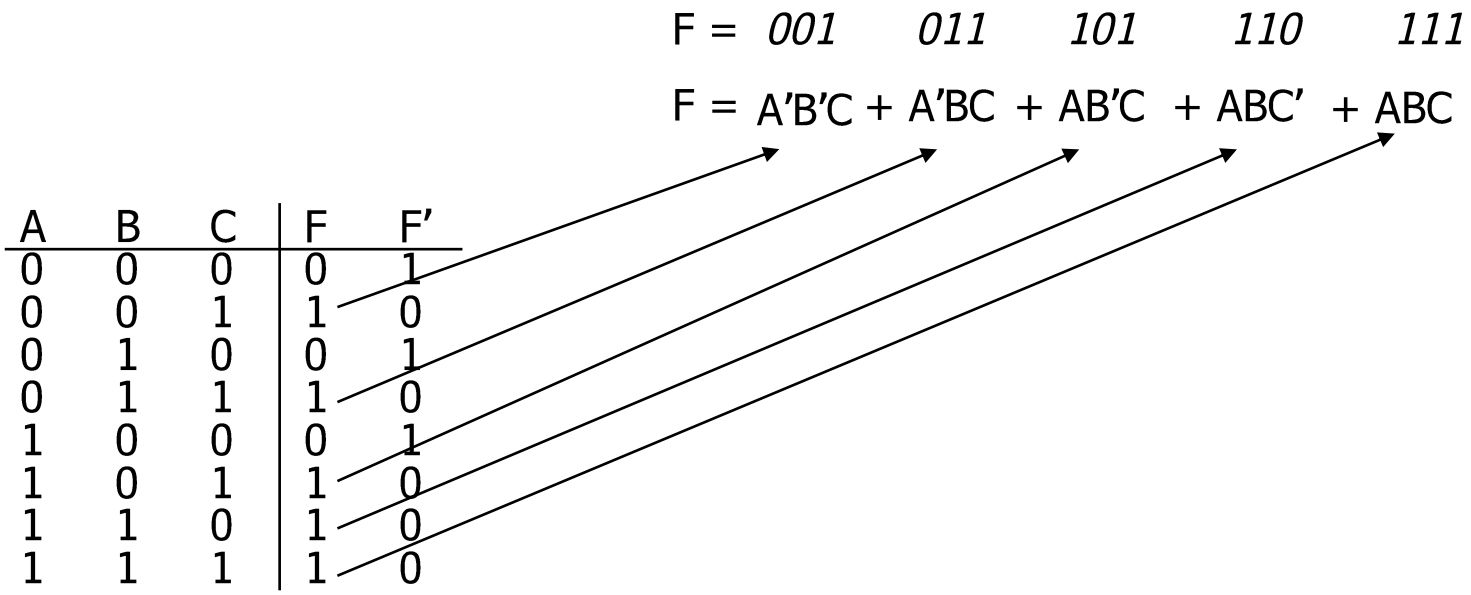
- Truth table is the unique signature of a Boolean function

- The same truth table can have many gate realizations
  - –we've seen this already
  - –depends on how good we are at Boolean simplification

- Canonical forms
  - standard forms for a Boolean expression
  - we all come up with the same expression

Sum-of-Products Canonical Form

- also known as Disjunctive Normal Form (DNF)
- also known as minterm expansion



Sum-of-Products Canonical Form

Product term (or minterm)

- **ANDed product of literals – input combination for which output is true**
- **each variable appears exactly once, true or inverted (but not both)**

A	B	C	minterms
0	0	0	A'B'C'
0	0	1	A'B'C
0	1	0	A'BC'
0	1	1	A'BC
1	0	0	AB'C'
1	0	1	AB'C
1	1	0	ABC'
1	1	1	ABC

F in canonical form:

$F(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC$

canonical form $\neq$ minimal form

$$
\begin{aligned}
F(A, B, C) &= A'B'C + A'BC + AB'C + ABC + ABC' \\
&= (A'B' + A'B + AB' + AB)C + ABC' \\
&= ((A' + A)(B' + B))C + ABC' \\
&= C + ABC' \\
&= ABC' + C \\
&= AB + C
\end{aligned}
$$

Product-of-Sums Canonical Form

- Also known as **Conjunctive Normal Form (CNF)**
- Also known as **maxterm expansion**

F = *000* *010* *100*

F = (A + B + C) (A + B’ + C) (A’ + B + C)

A	B	C	F	F’
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

s-o-p, p-o-s, and de Morgan's theorem

Complement of function in sum-of-products form:

- $F' = A'B'C' + A'BC' + AB'C'$

Complement again and apply de Morgan's and get the product-of-sums form:

- $(F')' = (A'B'C' + A'BC' + AB'C')'$
- $F = (A + B + C)\,(A + B' + C)\,(A' + B + C)$

Product-of-Sums Canonical Form

Sum term (or maxterm)

- **ORed sum of literals – input combination for which output is false**
- **each variable appears exactly once, true or inverted (but not both)**

A	B	C	maxterms
0	0	0	A+B+C
0	0	1	A+B+C’
0	1	0	A+B’+C
0	1	1	A+B’+C’
1	0	0	A’+B+C
1	0	1	A’+B+C’
1	1	0	A’+B’+C
1	1	1	A’+B’+C’

F in canonical form:

$$F(A, B, C) = (A + B + C)\,(A + B' + C)\,(A' + B + C)$$

canonical form $\neq$ minimal form

$$\begin{aligned} F(A, B, C) &= (A + B + C)\,(A + B' + C)\,(A' + B + C) \\ &= (A + B + C)\,(A + B' + C) \\ &\quad\ (A + B + C)\,(A' + B + C) \\ &= (A + C)\,(B + C) \end{aligned}$$