CSE 311: Foundations of Computing I

Section: Structural Induction and Regular Expressions Solutions

Structural Induction

(a) Recall the following definitions:

$$\begin{aligned}
\text{len}(\varepsilon) &= 0 \\
\text{len}(wa) &= \text{len}(w) + 1, \text{for } w \in \Sigma^*, a \in \Sigma
\end{aligned}$$

$$\begin{aligned}
x \bullet \varepsilon &= x, \text{for } x \in \Sigma^* \\
x \bullet wa &= (x \bullet w)a, \text{for } x \in \Sigma^*, a \in \Sigma
\end{aligned}$$

Consider the following recursive definition:

$$\begin{aligned}
\texttt{stutter}(\varepsilon) &= \varepsilon \\
\texttt{stutter}(wa) &= \texttt{stutter}(w) \bullet aa, \text{for } w \in \Sigma^*, a \in \Sigma
\end{aligned}$$

Prove that $\text{len}(\texttt{stutter}(w)) = 2\text{len}(w)$ for all $w \in \Sigma^*$.

Solution: Let $P(w)$ be "$\text{len}(\texttt{stutter}(w)) = 2\text{len}(w)$" for all $w \in \Sigma^*$. We go by structural induction.

Base Case. Note that $\text{len}(\texttt{stutter}(\varepsilon)) = \text{len}(\varepsilon) = 0 = 2 \cdot 0 = 2 \cdot \text{len}(\varepsilon)$. So, $P(\varepsilon)$ is true.

Induction Hypothesis. Suppose that $P(w)$ is true for some $w \in \Sigma^*$.

Induction Step. Note that

$$\begin{aligned}
\text{len}(\texttt{stutter}(wa)) &= \text{len}(\texttt{stutter}(w) \bullet aa) && [\text{By Definition of } \texttt{stutter}] \\
&= \text{len}((\texttt{stutter}(w) \bullet a)a) && [\text{By Definition of } \bullet] \\
&= \text{len}(\texttt{stutter}(w) \bullet a) + 1 && [\text{By Definition of len}] \\
&= \text{len}(\texttt{stutter}(w)a) + 1 && \\
&= \text{len}(\texttt{stutter}(w)) + 1 + 1 && [\text{By Definition of len}] \\
&= 2\text{len}(w) + 2 && [\text{By IH}] \\
&= 2(\text{len}(w) + 1) && \\
&= 2(\text{len}(wa)) && [\text{Definition of len}]
\end{aligned}$$

Thus, $P(w)$ is true for all $w \in \Sigma^*$ by structural induction.

Regular Expressions

(a) Write a regular expression that matches base 10 numbers (e.g., there should be no leading zeroes).

Solution:

$$(0 \cup (1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)(1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)^*)$$

(b) Write a regular expression that matches all base-3 numbers that are divisible by 3.

Solution:

$$(0 \cup 1 \cup 2)^* 0$$

(c) Write a regular expression that matches all binary strings that contain the substring "111", but not the substring "000".

Solution:

$$(01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon)111(01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon)$$