CSE 311: Foundations of Computing I

Section: FOL and Inference Solutions

Translate to Logic

Express each of these system specifications using predicate, quantifiers, and logical connectives.

(a) Every user has access to an electronic mailbox.

Solution: Let the domain be users and mailboxes. Let User(x) be "x is a user", let Mailbox(y) be "y is a mailbox", and let Access(x, y) be "x has access to y".

$$\forall x\ (\text{User}(x) \to (\exists y\ (\text{Mailbox}(y) \land \text{Access}(x, y))))$$

(b) The system mailbox can be accessed by everyone in the group if the file system is locked.

Solution: Let the domain be people in the group. Let Access(x, y) be "x has access to y". Let FileSystemLocked be the proposition "the file system is locked." Let SystemMailbox be the constant that is the system mailbox.

$$\text{FileSystemLocked} \to \forall x\ \text{Access}(x, \text{SystemMailbox})$$

(c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.

Solution: Let the domain be all applications. Let Firewall(x) be "x is the firewall", and let ProxyServer(x) be "x is the proxy server." Let Diagnostic(x) be "x is in a diagnostic state".

$$\forall x\ \forall y\ ((\text{Firewall}(x) \land \text{Diagnostic}(x)) \to (\text{ProxyServer}(y) \to \text{Diagnostic}(y))$$

(d) At least one router is functioning normally if the throughput is between 100kbps and 500 kbps and the proxy server is not in diagnostic mode.

Solution: Let the domain be all applications and routers. Let Router(x) be "x is a router", and let ProxyServer(x) be "x is the proxy server." Let Diagnostic(x) be "x is in a diagnostic state". Let ThroughputNormal be "the throughput is between 100kbps and 500 kbps". Let Functioning(y) be "y is functioning normally".

$$\forall x\ (\text{ThroughputNormal} \land (\text{ProxyServer}(x) \land \neg\text{Diagnostic}(x))) \to (\exists y\ \text{Router}(y) \land \text{Functioning}(y))$$

Translate to English

Translate these system specifications into English where $F(p)$ is "Printer p is out of service", $B(p)$ is "Printer p is busy", $L(j)$ is "Print job j is lost," and $Q(j)$ is "Print job j is queued". Let the domain be all printers.

(a) $\exists p\ (F(p) \land B(p)) \to \exists j\ L(j)$

Solution: If at least one printer is busy and out of service, then at least one job is lost.

(b) $(\forall p\ B(p)) \to (\exists j\ Q(j))$

Solution: If all printers are busy, then there is a queued job.

(c) $\exists j\ (Q(j) \wedge L(j)) \to \exists p\ F(p)$

Solution: If there is a queued job that is lost, then a printer is out of service.

(d) $(\forall p\ B(p) \wedge \forall j\ Q(j)) \to \exists j\ L(j)$

Solution: If all printers are busy and all jobs are queued, then there is some lost job.

Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other (or if they are equivalent).

(a) $\forall x\ \forall y\ P(x, y)$ $\qquad$ $\forall y\ \forall x\ P(x, y)$

Solution: These sentences are the same; switching universal quantifiers makes no difference.

(b) $\exists x\ \exists y\ P(x, y)$ $\qquad$ $\exists y\ \exists x\ P(x, y)$

Solution: These sentences are the same; switching existential quantifiers makes no difference.

(c) $\forall x\ \exists y\ P(x, y)$ $\qquad$ $\forall y\ \exists x\ P(x, y)$

Solution: These are only the same if P is symmetric (e.g. the order of the arguments doesn't matter). If the order of the arguments does matter, then these are different statements. For instance, if $P(x, y)$ is "$x < y$, then the first statement says "for every x, there is a corresponding y such that $x < y$, whereas the second says "for every y, there is a corresponding x such that $x < y$. In other words, in the first statement y is a function of x, and in the second x is a function of y.

(d) $\forall x\ \exists y\ P(x, y)$ $\qquad$ $\exists x\ \forall y\ P(x, y)$

Solution: These two statements are usually different.

Formal Proofs

For each of the following part, write *formal proofs*.

(a) Prove $\forall x\ (R(x) \wedge S(x))$ given $\forall x\ (P(x) \to (Q(x) \wedge S(x)))$, and $\forall x\ (P(x) \wedge R(x)$.

Solution:

1.	Let x be arbitrary.	
2.	$\forall x\ (P(x) \land R(x))$	[Given]
3.	$P(x) \land R(x)$	[Elim $\forall$: 2]
4.	$P(x)$	[Elim $\land$: 3]
5.	$R(x)$	[Elim $\land$: 3]
6.	$\forall x\ (P(x) \to (Q(x) \land S(x)))$	[Given]
7.	$P(x) \to (Q(x) \land S(x))$	[Elim $\forall$: 6]
8.	$Q(x) \land S(x)$	[MP: 4, 7]
9.	$S(x)$	[Elim $\land$: 8]
10.	$R(x) \land S(x)$	[Intro $\land$: 5, 9]
11.	$\forall x\ (R(x) \land S(x))$	[Intro $\forall$: 10]

(b) Prove $\exists x\ \neg R(x)$ given $\forall x\ (P(x) \lor Q(x))$, $\forall x\ (\neg Q(x) \lor S(x))$, $\forall x\ (R(x) \to \neg S(x))$, and $\exists x\ \neg P(x)$.

Solution:

1.	$\exists x\ \neg P(x)$	[Given]
2.	$\neg P(c)$	[Elim $\exists$: 1]
3.	$\forall x\ (P(x) \lor Q(x))$	[Given]
4.	$P(c) \lor Q(c)$	[Elim $\forall$: 3]
5.	$Q(c)$	[Elim $\lor$: 2, 4]
6.	$\forall x\ (\neg Q(x) \lor S(x))$	[Given]
7.	$\neg Q(c) \lor S(c)$	[Elim $\forall$: 6]
8.	$S(c)$	[Elim $\lor$: 5, 7]
9.	$\forall x\ (R(x) \to \neg S(x))$	[Given]
10.	$R(c) \to \neg S(c)$	[Elim $\forall$: 9]
11.	$\neg\neg S(c) \to \neg R(c)$	[Contrapositive: 10]
12.	$S(c) \to \neg R(c)$	[Double Negation: 11]
13.	$\neg R(c)$	[MP: 8, 12]
14.	$\exists x\ \neg R(x)$	[Intro $\exists$: 13]

English Proof

Prove that if a real number $x \neq 0$, then $x^2 + \frac{1}{x^2} \geq 2$.

Solution: Note that $(x^2 - 1)^2 \geq 0$, because all squares are at least 0. Distributing, we see that $x^4 + 1 \geq 2x^2$. Since $x \neq 0$, we can divide by x^2 to get $x^2 + \frac{1}{x^2} \geq 2$, which is what we were trying to prove.

Note: The first step may seem like "magic", but the way we generally solve these sorts of problems is by working backward and reversing the entire proof.

Primality Checking

When brute forcing if the number p is prime, you only need to check possible factors up to $\sqrt{p}$. In this problem, you'll prove why that is the case. Prove that if $n = ab$, then either a or b is at most $\sqrt{n}$.
(*Hint:* You want to prove an implication; so, start by assuming $n = ab$. Then, continue by writing out your assumption for contradiction.)

Solution: Suppose that $n = ab$. Suppose for contradiction that $a, b > \sqrt{n}$. It follows that $ab > \sqrt{n}\sqrt{n} = n$. We clearly can't have both $n = ab$ and $n < ab$; so, this is a contradiction. It follows that a or b is at most $\sqrt{n}$.