CSE 311: Foundations of Computing I

Section: Gates and Equivalences Solutions

Binary Addition

Just as a quick recall of binary, do the following operations. Then, convert your answers to base-10.

(a) $(101011)_2 + (1111)_2$

Solution: $(111010)_2 = 58$

(b) $(101011)_2 \oplus (1111)_2$

Solution: $(100100)_2 = 36$

(c) $(101011)_2 * (1111)_2$

Solution: $(1010000101)_2 = 645$

Equivalences

Prove that each of the following pairs of propositional formulae are equivalent using propositional equivalences.

(a) $p \leftrightarrow q$ $\qquad$ $(p \wedge q) \vee (\neg p \wedge \neg q)$

Solution:

$$
\begin{aligned}
p \leftrightarrow q \quad &\equiv \quad (p \rightarrow q) \wedge (q \rightarrow p) && \text{[iff is two implications]}\\
&\equiv \quad (\neg p \vee q) \wedge (\neg q \vee p) && \text{[Law of Implication]}\\
&\equiv \quad ((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p) && \text{[Distributivity]}\\
&\equiv \quad (\neg p \wedge \neg q) \vee (q \wedge \neg q) \vee ((p \wedge \neg p) \vee (p \wedge q)) && \text{[Distributivity]}\\
&\equiv \quad (\neg p \wedge \neg q) \vee (p \wedge q)) && \text{[Negation]}\\
&\equiv \quad (p \wedge q) \vee (\neg p \wedge \neg q) && \text{[Commutativity]}
\end{aligned}
$$

(b) $\neg p \rightarrow (q \rightarrow r)$ $\qquad$ $q \rightarrow (p \vee r)$

Solution:

$$
\begin{aligned}
\neg p \rightarrow (q \rightarrow r) \quad &\equiv \quad p \vee (\neg q \vee r) && \text{[Law of Implication]}\\
&\equiv \quad (p \vee \neg q) \vee r && \text{[Associativity]}\\
&\equiv \quad (\neg q \vee p) \vee r && \text{[Commutativity]}\\
&\equiv \quad \neg q \vee (p \vee r) && \text{[Associativity]}\\
&\equiv \quad q \rightarrow (p \vee r) && \text{[Law of Implication]}
\end{aligned}
$$

Tautologies

Prove that each of the following propositional formulae are tautologies by showing they are equivalent to T.

(a) $((p \to q) \wedge (q \to r)) \to (p \to r)$

Solution:

$$
\begin{aligned}
((p \to q) \wedge (q \to r)) \to (p \to r) &\equiv \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r) && \text{[Law Impl.]}\\
&\equiv ((p \wedge \neg q) \vee (q \wedge \neg r)) \vee (\neg p \vee r) && \text{[DeMorgan]}\\
&\equiv (((p \wedge \neg q) \vee (\neg p \vee r) \vee (q \wedge \neg r) && \text{[Associativity]}\\
&\equiv (((\neg p \vee r) \vee p) \wedge ((\neg p \vee r) \wedge \neg q)) \vee (q \wedge \neg r) && \text{[Distributivity]}\\
&\equiv (((p \vee \neg p) \vee r) \wedge ((\neg p \vee r) \vee \neg q)) \wedge (q \vee \neg r) && \text{[Assoc., Commut.]}\\
&\equiv ((\mathsf{T} \vee r) \wedge ((\neg p \vee r) \vee \neg q)) \vee (q \wedge \neg r) && \text{[Negation]}\\
&\equiv (\mathsf{T} \wedge ((\neg p \vee r) \vee \neg q)) \vee (q \wedge \neg r) && \text{[Domination]}\\
&\equiv ((\neg p \vee r) \vee \neg q) \vee (q \wedge \neg r) && \text{[Identity]}\\
&\equiv (q \vee ((\neg p \vee r) \vee \neg q) \wedge \neg r \vee ((\neg p \vee r) \vee \neg q)) && \text{[Distributivity]}\\
&\equiv ((q \vee \neg q) \vee (\neg p \vee r)) \wedge (\neg r \vee r \vee \neg p \vee \neg q) && \text{[Assoc., Comm.]}\\
&\equiv (\mathsf{T} \vee (\neg p \vee r)) \wedge (\mathsf{T} \vee \neg p \vee \neg q) && \text{[Negation]}\\
&\equiv \mathsf{T} \wedge \mathsf{T} && \text{[Domination]}\\
&\equiv \mathsf{T} && \text{[Identity]}
\end{aligned}
$$

(b) $(p \wedge q) \vee (p \wedge r) \to (q \vee r)$

Solution:

$$
\begin{aligned}
(p \wedge q) \vee (p \wedge r) \to (q \vee r) &\equiv (p \wedge (r \vee q)) \to (q \vee r) && \text{[Distrib.]}\\
&\equiv \neg(p \wedge (r \vee q)) \vee (q \vee r) && \text{[Law of Implication]}\\
&\equiv \neg(p \vee \neg(r \vee q)) \vee (q \vee r) && \text{[DeMorgan]}\\
&\equiv \neg p \vee (\neg(r \vee q) \vee (r \vee q)) && \text{[Assoc., Comm.]}\\
&\equiv \neg p \vee \mathsf{T} && \text{[Negation]}\\
&\equiv \mathsf{T} && \text{[Identity]}
\end{aligned}
$$

(c) $(p \wedge q) \vee (\neg p \wedge q) \vee \neg q$

Solution:

$$
\begin{aligned}
(p \wedge q) \vee (\neg p \wedge q) \vee \neg q &\equiv (q \wedge (p \vee \neg p)) \vee \neg q && \text{[Comm., Assoc., Distrib.]}\\
&\equiv (q \wedge \mathsf{T}) \vee \neg q && \text{[Negation]}\\
&\equiv q \vee \neg q && \text{[Identity]}\\
&\equiv \mathsf{T} && \text{[Negation]}
\end{aligned}
$$

Non-equivalence

Prove that each of the following pairs of propositional formulae are not equivalent by finding an input they differ on.

(a) $p \to q$ $\qquad\qquad$ $q \to p$

Solution: When $p =$ T and $q =$ F, then $p \to q \equiv$ F, but $q \to p \equiv$ T.

(b) $(p \to q) \to r$ $\qquad\qquad$ $p \to (q \to r)$

Solution: When $p =$ F, $q =$ F, and $r =$ F, then $(p \to q) \to r \equiv$ F, but $p \to (q \to r) \equiv$ T.

Convert To A Circuit

(a) $\neg((p \vee q) \wedge (p \vee r)) \vee (q \vee r)$

Solution: Solution omitted due to effort in typesetting. Please come to office hours if you want to see it.

Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and laws of boolean algebra.

(a) $\neg p \vee (\neg q \vee (p \wedge q))$

Solution: First, we replace $\neg, \vee$, and $\wedge$. This gives us $p' + q' + pq$; note that the parentheses are not necessary in boolean algebra, because the operations are all commutative and associative. We can use DeMorgan's laws to get the slightly simpler $(pq)' + pq$. Then, we can use complementarity to get 1. (Note that this is another way of saying the formula is a tautology.)

(b) $\neg(p \vee (q \wedge p))$

Solution: First, we use DeMorgan's laws to push the negation through. This gives us $\neg p \wedge (\neg q \vee \neg p)$. Now, we convert to $+$ and $\cdot$ which results in $p'(q' + p')$. Using distribution, we get $p'q' + p'p'$, and by idempotency, we get $p'q' + p'$. Then, we can use identity to get $p'q' + p'1$. Then, factoring out p' gives us $p'(q' + 1)$. Using null gives us $p'1$, and identity gives us p'