INSTRUCTIONS:

- You have 50 minutes to complete the exam.
- The exam is closed book and closed notes. You may not use cell phones or calculators.
- All answers you want graded should be written on the exam paper.
- If you need extra space, use the back of a page.
- The problems are of varying difficulty.
- If you get stuck on a problem, move on and come back to it later.
- It is to your advantage to read all the problems before beginning the exam.

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</table>
1. To Logic... or Not To Logic [20 points]

(a) (5 points) Choose a meaning of $P(x, y, z)$ such that $\forall x \exists y \forall z P(x, y, z)$ is false, but $\forall x \forall y \exists z P(x, y, z)$ is true.

Solution: Let the domain be $\mathbb{N}$. Let $P(x, y, z)$ be “$x \geq 0 \land y \geq z$”.

Then, the first statement is false, because, while $x \geq 0$ for everything in the domain, there is no largest number in the domain. However, the second statement is true, because $x \geq 0$ and $z = y$ satisfies the second part.

(b) (5 points) In the domain of integers, using any standard mathematical notation (but no new predicates), define Prime($x$) to mean “$x$ is prime”.

Solution: Prime($x$) $\equiv x \geq 2 \land \forall y ((1 \leq y \leq x \land y \mid x) \rightarrow (y = x \lor y = 1))$

Let the predicates $D(x, y)$ mean “team $x$ defeated team $y$” and $P(x, y)$ mean “team $x$ has played team $y$.” Give quantified formulas with the following meanings:

(c) (5 points) Every team has lost at least one game.

Solution: $\forall x \exists y D(y, x)$

(d) (5 points) There is a team that has beaten every team it has played.

Solution: $\exists x \forall y (P(x, y) \rightarrow D(x, y))$
2. Obvious Induction Problem [20 points]
Prove for all $n \in \mathbb{N}$ that the following identity is true:

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

where $x \in \mathbb{R}, x \neq 1$.

Solution: Let $P(n)$ be the statement “$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$” for all $n \in \mathbb{N}$.

We go by induction on $n$.

Base Case: When $n = 0$, $P(0)$ is true, because

$$\sum_{i=0}^{0} x^i = x^0 = 1 = \frac{1 - x^1}{1 - x}$$

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step: We see that

$$\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{k} x^i + x^{k+1}$$

[Taking out the last term]

$$= \frac{1 - x^{k+1}}{1 - x} + x^{k+1}$$

[By the IH]

$$= \frac{(1 - x^{k+1}) + (1 - x)x^{k+1}}{1 - x}$$

[Algebra]

$$= \frac{1 - x^{k+2}}{1 - x}$$

[Simplifying]

which is what we wanted to show in the induction step. Thus, we have proven $P(n)$ for all $n \in \mathbb{N}$ by induction.
3. 311 is Prime! [15 points]
Find all solutions in the range $0 \leq x < 311$ to the modular equation:

$$12x \equiv 5 \pmod{311}$$

Solution: First, we find the multiplicative inverse of 12 modulo 311. Note that it exists, because $\gcd(311, 12) = \gcd(12, 11) = \gcd(11, 1) = \gcd(1, 0) = 1$.
Now, we do the Extended Euclidean Algorithm:

\[
\begin{align*}
311 &= 12 \cdot 25 + 11 \\
12 &= 11 \cdot 1 + 1
\end{align*}
\]

Now, backwards substituting:

\[
1 = 12 - 11 \cdot 1 = 12 - (311 - 12 \cdot 25) \cdot 1 = 311 \cdot 1 + 12 \cdot 26
\]

So, the multiplicative inverse of 12 modulo 311 is 26.
Now, we have the modular equation $12(26) \equiv 1 \pmod{311}$. Multiplying both sides by 5, we get:

$$12(26 \cdot 5) \equiv 5 \pmod{311} \rightarrow 12(130) \equiv 5 \pmod{311}$$

So, $x = 130$. 

4. **Even Circuits Are Fun** [25 points]
The function multiple-of-three takes in two inputs: $(x_1x_0)_2$ and outputs 1 iff $3 \mid (x_1x_0)_2$.

(a) (5 points) Draw a table of values (e.g. a truth table) for multiple-of-three.

\[ \begin{array}{ccc}
 x_1 & x_0 & \text{multiple-of-three} \\
 0 & 0 & 1 \\
 0 & 1 & 0 \\
 1 & 0 & 0 \\
 1 & 1 & 1 \\
\end{array} \]

(b) (5 points) Write multiple-of-three as a sum-of-products.

\[ \text{Solution:} \quad \text{multiple-of-three} = (x_1'x_0') + (x_1x_0) \]

(c) (5 points) Write multiple-of-three as a product-of-sums.

\[ \text{Solution:} \quad \text{multiple-of-three} = (x_1 + x_0')(x_1' + x_0) \]

(d) (5 points) Write multiple-of-three as a simplified expression (don’t bother explaining what rules you’re using).

\[ \text{Solution:} \quad \text{multiple-of-three} = (x_1 + x_0)' + (x_1x_0) \]

(e) (5 points) Draw a boolean circuit to implement multiple-of-three.

\[ \text{Solution:} \quad \text{Omitted.} \]
5. Irrationally Rational [10 points]
Recall the definition of irrational is that a number is not rational, and that

\[
\text{Rational}(x) \equiv \exists p \exists q \ x = \frac{p}{q} \land \text{Integer}(p) \land \text{Integer}(q) \land q \neq 0
\]

For this question, you may assume that \( \pi \) is irrational. Disprove that if \( x \) and \( y \) are irrational, then \( x + y \) is irrational.

\textbf{Solution:} Note that \( \pi \) is irrational, and multiplying by \(-1\) maintains irrationality (because if it didn’t, then we could find \( p, q \) by multiplying by \(-1\), getting \( p, q \), and choosing \(-p\) and \( q\)). Finally, note that \( \pi + (-\pi) = 0 \), which is rational.

6. Rationally Irrational [10 points]
Recall the definition of irrational is that a number is not rational, and that

\[
\text{Rational}(x) \equiv \exists p \exists q \ x = \frac{p}{q} \land \text{Integer}(p) \land \text{Integer}(q) \land q \neq 0
\]

Prove that if \( x \) and \( y \) are rational and \( x \neq 7 \), then \( \frac{y^2}{x-7} \) is rational.

\textbf{Solution:} Suppose \( x, y \) are rational and \( x \neq 7 \). We previously proved that if \( a \) and \( b \) are rational, then \( ab \) is rational. So, \( y^2 \) is rational, which means we have integers \( p, q, q \neq 0 \) such that \( y^2 = p/q \). Furthermore, since \( x - 7 \neq 0 \), \( \frac{1}{x-7} \) is rational, because \( a = 1, b = x - 7 \) are integers with \( b \neq 0 \) such that \( \frac{1}{x-7} = a/b \). Finally, note that \( \frac{y^2}{x-7} = y^2 \frac{1}{x-7} = (p/q)(a/b) = (pa)/(qb) \). Since \( q \) and \( b \) are both non-zero, \( qb \) is non-zero. Also, \( pq \) and \( qb \) are integers. Thus, it follows that \( \frac{y^2}{x-7} \) is rational.
Identity

- \( p \land T \equiv p \)
- \( p \lor F \equiv p \)

Domination

- \( p \lor T \equiv T \)
- \( p \land F \equiv F \)

Idempotency

- \( p \lor p \equiv p \)
- \( p \land p \equiv p \)

Commutativity

- \( p \lor q \equiv q \lor p \)
- \( p \land q \equiv q \land p \)

Associativity

- \( (p \lor q) \lor r \equiv p \lor (q \lor r) \)
- \( (p \land q) \land r \equiv (p \land q) \land r \)

Distributivity

- \( p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \)
- \( p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \)

Absorption

- \( p \lor (p \land q) \equiv p \)
- \( p \land (p \lor q) \equiv p \)

Negation

- \( p \lor \neg p \equiv T \)
- \( p \land \neg p \equiv F \)

DeMorgan’s Laws

- \( \neg (p \lor q) \equiv \neg p \land \neg q \)
- \( \neg (p \land q) \equiv \neg p \lor \neg q \)

Double Negation

- \( \neg \neg p \equiv p \)

Law of Implication

- \( p \rightarrow q \equiv \neg p \lor q \)

Contrapositive

- \( p \rightarrow q \equiv \neg q \rightarrow \neg p \)
## Inference Rules

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<td>$\therefore q$</td>
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<td>$p \Rightarrow q$</td>
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<td><strong>Elim $\land$</strong></td>
<td>$p \land q$</td>
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<td><strong>Intro $\land$</strong></td>
<td>$p, q$</td>
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<td><strong>Elim $\lor$</strong></td>
<td>$p \lor q, \neg p$</td>
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<td><strong>Intro $\lor$</strong></td>
<td>$p$</td>
<td>$\therefore p \lor q, q \lor p$</td>
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<td><strong>Excluded Middle</strong></td>
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<td><strong>Elim $\forall$</strong></td>
<td>$\forall x P(x)$</td>
<td>$\therefore P(a)$ for any $a$</td>
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<td><strong>Intro $\forall$</strong></td>
<td>Let $a$ be an arbitrary...</td>
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<td><strong>Elim $\exists$</strong></td>
<td>$\exists x P(x)$</td>
<td>$\therefore P(c)$ for some special $c$</td>
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<td><strong>Intro $\exists$</strong></td>
<td>$P(c)$ for some $c$</td>
<td>$\therefore \exists x P(x)$</td>
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