CSE 311: Foundations of Computing I — Autumn 2014

Practice Final: Section X — YY ZZ

Name: ____

UW ID: ____

Instructions:

- Closed book, closed notes, no cell phones, no calculators.
- You have **110 minutes** to complete the exam.
- Answer all problems on the exam paper.
- If you need extra space use the back of a page.
- Problems are not of equal difficulty; if you get stuck on a problem, move on.
- You may tear off the last two pages of equivalence and inference rules. These must be handed in at the end but will not be graded.
- It may be to your advantage to read all the problems before beginning the exam.

Score Table Here

1. [? points]
Let $\Sigma = \{0, 1\}$. Prove that the language $L = \{x \in \Sigma^* : \#_0(x) < \#_1(x)\}$ is irregular.

Solution: Let D be an arbitrary DFA. Consider $S = \{0^n : n \geq 0\}$. Since S is infinite and D has finitely many states, we know $0^i \in S$ and $0^j \in S$ both end in the same state for some $i < j$. Append 1^j to both strings to get:

$a = 0^i 1^j$ Note that $a \in L$, because $i < j$ and $0^i 1^j \in \Sigma^*$.

$b = 0^j 1^j$ Note that $b \notin L$, because $j \not< j$.

Since a and b both end in the same state, and that state cannot both be an accept and reject state, D cannot recognize L. Since D was arbitrary, no DFA recognizes L; so, L is irregular.

2. [? points]
Define

$$T(n) = \begin{cases} n & \text{if } n = 0, 1 \\ 4T\left(\lfloor \frac{n}{2} \rfloor\right) + n & \text{otherwise} \end{cases}$$

Prove that $T(n) \leq n^3$ for $n \geq 3$.

Solution: We go by strong induction on n. Let $P(n)$ be "$T(n) \leq n^3$" for $n \in \mathbb{N}$.

Base Cases. When $n = 3$, $T(3) = 4T\left(\lfloor \frac{3}{2} \rfloor\right) + 3 = 4T(1) + 3 = 7 \leq 27 = 3^3$.
When $n = 4$, $T(4) = 4T\left(\left\lfloor \frac{4}{2} \right\rfloor\right) + 4 = 4T(2) + 4 = 27 \leq 64 = 4^3$.
When $n = 5$, $T(5) = 4T\left(\left\lfloor \frac{5}{2} \right\rfloor\right) + 5 = 4T(2) + 5 = 28 \leq 4^4$.

Induction Hypothesis. Suppose $P(3) \wedge P(4) \wedge \cdots \wedge P(k)$ for some $k \geq 5$.

Induction Step. We want to prove $P(k+1)$: Note that

$$\begin{aligned}
T(k+1) &= 4T\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right) + k + 1, && \text{because } k+1 \geq 2. \\
&\leq 4\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right)^3 + k + 1, && \text{by IH.} \\
&\leq 4\left(\frac{k+1}{2}\right)^3 + k + 1, && \text{by def of floor.} \\
&= 4\left(\frac{(k+1)^3}{2^3}\right) + k + 1, && \text{by algebra.} \\
&= \frac{(k+1)^3}{2} + k + 1, && \text{by algebra.} \\
&= \frac{(k+1)((k+1)^2 + 2)}{2}, && \text{by algebra.} \\
&\leq \frac{(k+1)((k+1)^2 + (k+1)^2)}{2}, && \text{because } (k+1)^2 \geq 2. \\
&= (k+1)^3, && \text{by algebra}
\end{aligned}$$

Thus, since the base case and induction step hold, the $P(n)$ is true for $n \geq 3$.

3. [? points]
Let $\Sigma = \{0, 1, 2\}$.
Consider $L = \{w \in \Sigma^*$: Every 1 in the string has at least one 0 before and after it$\}$.

a) Give a regular expression that represents A.

Solution: $(0 \cup 2)^*(0(0 \cup 1 \cup 2)^*0)^*(0 \cup 2)^*$

b) Give a DFA that recognizes A.

Solution: Omitted.

c) Give a CFG that generates A.

Solution:

$$S \rightarrow \varepsilon \mid 0\mathbf{S} \mid 2\mathbf{S} \mid 0\mathbf{ST}$$
$$T \rightarrow 1\mathbf{R}0\mathbf{S}$$
$$R \rightarrow \varepsilon \mid 0\mathbf{R} \mid 1\mathbf{R} \mid 2\mathbf{R}$$

4. [? points]
Consider the following CFG: **S** $\rightarrow$ **SS** | **S**1 | **S**01. Another way of writing the recursive definition of this set, Q, is as follows:

- $\varepsilon \in Q$
- If $S \in Q$, then $S1 \in Q$ and $S01 \in Q$
- If $S, T \in Q$, then $ST \in Q$.

Prove, by structural induction that if $w \in Q$, then w has at least as many 1's as 0's.

Solution: We go by structal induction on w. Let $P(w)$ be "$\#_0(w) \leq \#_1(w)$" for $w \in \Sigma^*$.

Base Case. When $w = \varepsilon$, note that $\#_0(w) = 0 = \#_1(w)$. So, the claim is true.

Induction Hypothesis. Suppose $P(w), P(v)$ are true for some w, v generated by the grammar.

Induction Step 1. Note that $\#_0(w1) = \#_0(w) \leq \#_1(w)+1 = \#_1(w1)$ by IH, and $\#_0(w01) = \#_0(w) + 1 \leq \#_1(w) + 1 = \#_1(w01)$ by IH.

Induction Step 2. Note that $\#_0(wv) = \#_0(w) + \#_0(v) \leq \#_1(w) + \#_1(v)$ by IH.

Since the claim is true for all recursive rules, the claim is true for all strings generated by the grammar.

5. [? points]
For each of the following answer True or False and give a short explanation of your answer.

- Any subset of a regular language is also regular.

  Solution: False. Consider $\{0,1\}^*$ and $\{0^n 1^n : n \geq 0\}$. Note that the first is regular and the second is irregular, but the second is a subset of the first.

- The set of programs that loop forever on at least one input is decidable.

  Solution: False. If we could solve this problem, we could solve HaltNoInput. Intuitively, a program that solves this problem would have to try all inputs, but, since the program might infinte loop on some of them, it won't be able to.

- If $\mathbb{R} \subseteq A$ for some set A, then A is uncountable.

  Solution: True. Diagonalization would still work; alternatively, if A were countable, then we could find an onto function between $\mathbb{N}$ and $\mathbb{R}$ by skipping all the elements in A that aren't in $\mathbb{R}$.

- If the domain of discourse is people, the logical statement

  $$\exists x\ (P(x) \rightarrow \forall y\ (x \neq y \rightarrow \neg P(y))$$

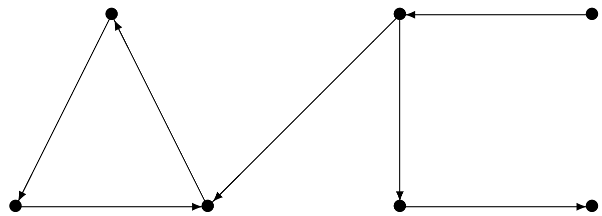
  can be correctly translated as "There exists a unique person who has property P".

  Solution: False. Any x for which $P(x)$ is false makes the entire statement true. This is not the same as there existing a unique person with property P.

- $\exists x\ (\forall y\ P(x,y)) \rightarrow \forall y\ (\exists x\ P(x,y))$ is true regardless of what predicate P is.

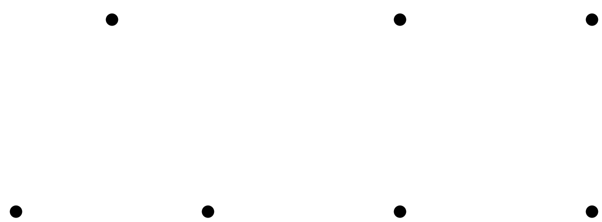
  Solution: True. The left part of the implication is saying that there is a single x that works for all y; the right one is saying that for every y, we can find an x that depends on it, but the single x that works for everything will still work.

6. [? points]
The following is the graph of a binary relation R.



a) Draw the transitive-reflexive closure of R.

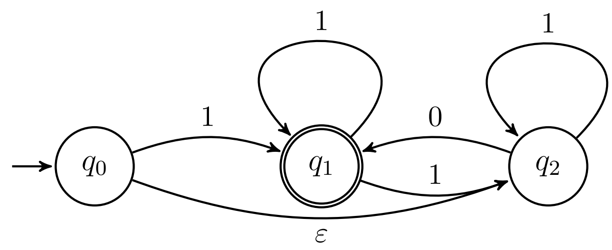
Solution: Omitted.



b) Let $S = \{(X, Y) : X, Y \in \mathcal{P}(\mathbb{N}) \land X \subseteq Y\}$.
Recall that R is antisymmetric iff $((a, b) \in R \land a \neq b) \rightarrow (b, a) \notin R$.
Prove that S is antisymmetric.

Solution: Suppose $(a, b) \in S$ and $a \neq b$. Then, by definition of S, $a \subset b$ and there is some $x \in b$ where $x \notin a$ (since they aren't equal). Then, $(b, a) \notin S$, because $b \nsubseteq a$, because $x \in b$ and $x \notin a$. So, S is antisymmetric.

7. [? points]
Convert the following NFA into a DFA using the algorithm from lecture.



Solution: Omitted.

8. [? points]
Let $\Sigma = \{0, 1, 2\}$. Construct a DFA that recognizes exactly strings with a 0 in all positions i where i mod $3 = 0$.

Solution: Omitted.