CSE 311 Foundations of Computing I

Lecture 28
Computability Review, Course Summary
Spring 2013

Announcements

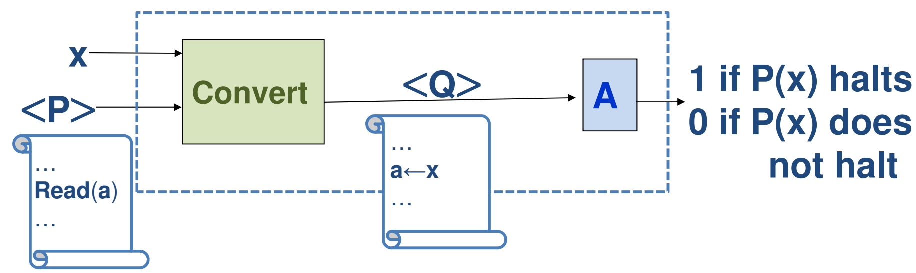
- Hand in Homework 8 now
- Pick up all old homework now
- Review session
  - Sunday, June 9, 4 pm, EEB 125
  - List of Final Exam Topics and sampling of some typical kinds of exam questions on the web
  - Bring your questions to the review session!
- Final exam
  - Monday, June 10, 2:30-4:20 pm, MGH 389

What makes the Halting Problem and related problems hard?

- Figuring out in some finite time something about what might happen in the infinite # of steps that a general computation might take
  - e.g. having **H(<P>,x)** output **0** when **P** doesn't halt on **x**
- The following program does exist and won't generate any contradiction
  - Function **V(x)**:
    - if **U(x,x)** halts then
      - **while** (true); /* loop forever */
    - else
      - **no-op**; /* do nothing and halt */
    - endif

The "Always Halting" problem

Suppose we had a TM **A** for the Always Halting problem

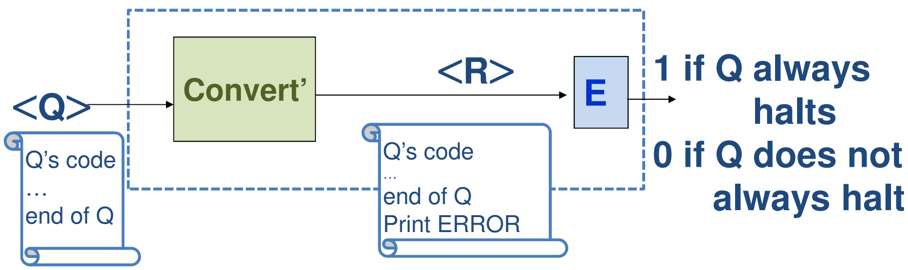


We designed **<Q>** based on **<P>** and **x** so that:
<Q> always halts ⇔ **<P>** halts on input **x**

So.. if **A** exists then we get a program **H** for the ordinary halting problem, which we know can't exist so **A** can't exist

The “Always ERROR” problem

Suppose we had a TM **E** for the ERROR problem



We designed <**R**> based on <**Q**> so that:

<**R**> always prints “ERROR” ⇔ <**Q**> always halts

So.. if **E** exists then we get a program **A** for the Always Halts problem, which we know can’t exist so **E** can’t exist

A general phenomenon: Can’t tell a book by its cover

Rice’s Theorem: In general there is no way to tell anything about the input/output (**I**/**O**) behavior of a program **P** just given its code <**P**>!

Quick lessons

- Don’t rely on the idea of improved compilers and programming languages to eliminate major programming errors
  - truly safe languages can’t possibly do general computation
- Document your code!!!!
  - there is no way you can expect someone else to figure out what your program does with just your code ....since....in general it is provably impossible to do this!

CSE 311 Foundations of Computing I

Spring 2013

Course Summary

About the course

- From the CSE catalog:
  - **CSE 311 Foundations of Computing I (4)** Examines fundamentals of logic, set theory, induction, and algebraic structures with applications to computing; finite state machines; and limits of computability. Prerequisite: CSE 143; either MATH 126 or MATH 136.
- What this course is about:
  - Foundational structures for the practice of computer science and engineering

Propositional Logic

- Statements with truth values
  - The Washington State flag is red
  - It snowed in Whistler, BC on January 4, 2011.
  - Rick Perry won the Iowa straw poll
  - Space aliens landed in Roswell, New Mexico
  - If n is an integer greater than two, then the equation $a^n + b^n = c^n$ has no solutions in non-zero integers a, b, and c.
- Propositional variables: *p, q, r, s, . . .*
- Truth values: **T** for true, **F** for false
- Compound propositions

Negation (not)	$\neg p$
Conjunction (and)	$p \wedge q$
Disjunction (or)	$p \vee q$
Exclusive or	$p \oplus q$
Implication	$p \rightarrow q$
Biconditional	$p \leftrightarrow q$

English and Logic

- You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old
  - *q*: you can ride the roller coaster
  - *r*: you are under 4 feet tall
  - *s*: you are older than 16

$$\boxed{(r \wedge \neg s) \rightarrow \neg q}$$

Logical equivalence

- Terminology: A compound proposition is a
  - *Tautology* if it is always true
  - *Contradiction* if it is always false
  - *Contingency* if it can be either true or false

$p \vee \neg p$

$p \oplus p$

$(p \rightarrow q) \wedge p$

$(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

Logical Equivalence

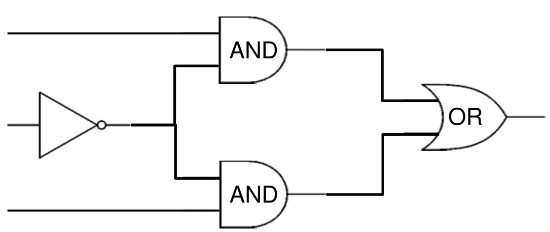
- p and q are *logically equivalent* iff $p \leftrightarrow q$ is a tautology
- The notation $p \equiv q$ denotes p and q are logically equivalent
- De Morgan's Laws:

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg (p \vee q) \equiv \neg p \wedge \neg q$$

Digital Circuits

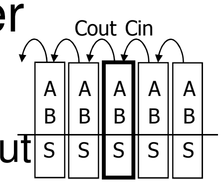
- Computing with logic
  - **T** corresponds to 1 or "high" voltage
  - **F** corresponds to 0 or "low" voltage
- Gates
  - Take inputs and produce outputs
    - Functions
  - Several kinds of gates
  - Correspond to propositional connectives
    - Only symmetric ones (order of inputs irrelevant)

Combinational Logic Circuits



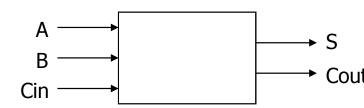
Wires can send one value to multiple gates

A simple example: 1-bit binary adder



- Inputs: A, B, Carry-in
- Outputs: Sum, Carry-out

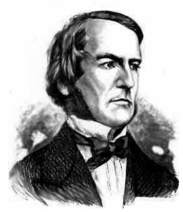
A	B	Cin	Cout	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



Cout = B Cin + A Cin + A B

S = A′ B′ Cin + A′ B Cin′ + A B′ Cin′ + A B Cin
= A′ (B′ Cin + B Cin′) + A (B′ Cin′ + B Cin)
= A′ Z + A Z′
= A xor Z = A xor (B xor Cin)

Boolean algebra



George Boole – 1854

- An algebraic structure consists of
  - a set of elements B
  - binary operations { + , • }
  - and a unary operation { ’ }
  - such that the following axioms hold:

1. the set B contains at least two elements: a, b		
2. closure:	a + b is in B	a • b is in B
3. commutativity:	a + b = b + a	a • b = b • a
4. associativity:	a + (b + c) = (a + b) + c	a • (b • c) = (a • b) • c
5. identity:	a + 0 = a	a • 1 = a
6. distributivity:	a + (b • c) = (a + b) • (a + c)	a • (b + c) = (a • b) + (a • c)
7. complementarity:	a + a’ = 1	a • a’ = 0

Sum-of-products canonical forms

- Also known as disjunctive normal form
- Also known as minterm expansion

F = *001* *011* *101* *110* *111*

F = A′B′C + A′BC + AB′C + ABC′ + ABC

A	B	C	F	F′
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

F′ = A′B′C′ + A′BC′ + AB′C′

Predicate Calculus

- *Predicate* or *Propositional Function*
  - A function that returns a truth value
- “*x* is a cat”
- “student *x* has taken course *y*”
- “$x > y$”
- $\forall x\ P(x)$: $P(x)$ is true for every *x* in the domain
- $\exists x\ P(x)$: There is an *x* in the domain for which $P(x)$ is true

Statements with quantifiers

Domain:
Positive Integers

Even(*x*)
Odd(*x*)
Prime(*x*)
Greater(*x*,*y*)
Equal(*x*,*y*)

- $\forall x\ (\mathrm{Even}(x) \vee \mathrm{Odd}(x))$
- $\exists x\ (\mathrm{Even}(x) \wedge \mathrm{Prime}(x))$
- $\forall x\ \exists y\ (\mathrm{Greater}(y, x) \wedge \mathrm{Prime}(y))$
- $\forall x\ (\mathrm{Prime}(x) \rightarrow (\mathrm{Equal}(x, 2) \vee \mathrm{Odd}(x))$
- $\exists x\ \exists y (\mathrm{Equal}(x, y + 2) \wedge \mathrm{Prime}(x) \wedge \mathrm{Prime}(y))$

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

Simple Propositional Inference Rules

- Excluded middle $\dfrac{}{\therefore\ p \lor \neg p}$
- Two inference rules per binary connective one to eliminate it, one to introduce it.

$$\frac{p \land q}{\therefore\ p,\ q} \qquad \frac{p,\ q}{\therefore\ p \land q}$$

$$\frac{p \lor q,\ \neg p}{\therefore\ q} \qquad \frac{p}{\therefore\ p \lor q,\ q \lor p}$$

$$\frac{p,\ p \to q}{\therefore\ q} \qquad \frac{p \Rightarrow q}{\therefore\ p \to q}$$

Direct Proof Rule

Inference Rules for Quantifiers

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists\, x\ P(x)} \qquad \frac{\forall\, x\ P(x)}{\therefore\ P(a) \text{ for any } a}$$

$$\frac{\text{“Let } a \text{ be anything”} \ldots P(a)}{\therefore\ \forall\, x\ P(x)} \qquad \frac{\exists\, x\ P(x)}{\therefore\ P(c) \text{ for some special } c}$$

Even and Odd

$\text{Even}(x) \equiv \exists y\ \ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ \ (x=2y+1)$
Domain: Integers

- Prove: “The square of every odd number is odd”
  English proof of: $\forall x\ (\text{Odd}(x) \to \text{Odd}(x^2))$

Let x be an odd number.
Then $x=2k+1$ for some integer k (depending on x)
Therefore $x^2=(2k+1)^2= 4k^2+4k+1=2(2k^2+2k)+1$.
Since $2k^2+2k$ is an integer, x^2 is odd.

Characteristic vectors

- Let U = {1, . . ., 10}, represent the set {1,3,4,8,9} with

  1011000110

- Bit operations:
  - 0110110100 $\vee$ 0011010110 = 0111110110
- **`ls -l`**

```
drwxr-xr-x ... Documents/
-rw-r--r-- ... file1
```


One-time pad

- Alice and Bob privately share random n-bit vector K
  - Eve does not know K
- Later, Alice has n-bit message m to send to Bob
  - Alice computes C = m $\oplus$ K
  - Alice sends C to Bob
  - Bob computes m = C $\oplus$ K which is (m $\oplus$ K) $\oplus$ K
- Eve cannot figure out m from C unless she can guess K

Arithmetic mod 7

- a $+_7$ b = (a + b) mod 7
- a $\times_7$ b = (a $\times$ b) mod 7

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Division Theorem

Let a be an integer and d a positive integer. Then there are *unique* integers q and r, with $0 \leq r < d$, such that $a = dq + r$.

$q = a$ **div** d $\qquad$ $r = a$ **mod** d

Modular Arithmetic

Let a and b be integers, and m be a positive integer. We say a *is congruent to b modulo m* if m divides a – b. We use the notation a ≡ b (mod m) to indicate that a is congruent to b modulo m.

Let a and b be integers, and let m be a positive integer. Then a ≡ b (mod m) if and only if a mod m = b mod m.

Let m be a positive integer. If a ≡ b (mod m) and c ≡ d (mod m), then
a + c ≡ b + d (mod m) and
ac ≡ bd (mod m)

Let a and b be integers, and let m be a positive integer. Then a ≡ b (mod m) if and only if
a mod m = b mod m.

Integer representation

Signed integer representation

Suppose $-2^{n-1} < x < 2^{n-1}$
First bit as the sign, n-1 bits for the value

99: 0110 0011, -18: 1001 0010

Two's complement representation

Suppose $0 \le x < 2^{n-1}$,
x is represented by the binary representation of x
-x is represented by the binary representation of 2^n-x

99: 0110 0011, -18: 1110 1110

Hashing

- Map values from a large domain, 0…M-1 in a much smaller domain, 0…n-1
- Index lookup
- Test for equality
- Hash(x) = x mod p
  - (or Hash(x) = (ax + b) mod p)
- Often want the hash function to depend on all of the bits of the data
  - Collision management

Modular Exponentiation

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1	1	1	1	1	1	1
2	2	4	1	2	4	1
3	3	2	6	4	5	1
4	4	2	1	4	2	1
5	5	4	6	2	3	1
6	6	1	6	1	6	1

Arithmetic mod 7

Fast exponentiation
Repeated Squaring

```
namespace _311ConsoleApp {
    class Program {
        static void Main(string[] args) {
            FastExp(2, 16, 10000);
            System.Console.ReadLine();
        }

        static int FastExp(int x, int n, int modulus) {
            long v = (long)x;
            int exp = 1;
            for (int i = 1; i <= n; i++) {
                v = (v * v) % modulus;
                exp = exp + exp;
                System.Console.WriteLine("i: " + i
                    + "  exponent: " + exp + " v: " + v);
            }
            return (int)v;
        }
    }
}
```

file:///C:/Users/Richard/D...

```
i: 1  exponent: 2 v: 4
i: 2  exponent: 4 v: 16
i: 3  exponent: 8 v: 256
i: 4  exponent: 16 v: 5536
i: 5  exponent: 32 v: 7296
i: 6  exponent: 64 v: 1616
i: 7  exponent: 128 v: 1456
i: 8  exponent: 256 v: 9936
i: 9  exponent: 512 v: 4096
i: 10  exponent: 1024 v: 7216
i: 11  exponent: 2048 v: 656
i: 12  exponent: 4096 v: 336
i: 13  exponent: 8192 v: 2896
i: 14  exponent: 16384 v: 6816
i: 15  exponent: 32768 v: 7856
i: 16  exponent: 65536 v: 6736
```


Primality

An integer *p* greater than 1 is called *prime* if the only positive factors of *p* are 1 and *p*.

A positive integer that is greater than 1 and is not prime is called composite.

Fundamental Theorem of Arithmetic: Every positive integer greater than 1 has a unique prime factorization

GCD and Factoring

$a = 2^3 \bullet 3 \bullet 5^2 \bullet 7 \bullet 11 = 46{,}200$

$b = 2 \bullet 3^2 \bullet 5^3 \bullet 7 \bullet 13 = 204{,}750$

$$GCD(a, b) = 2^{\min(3,1)} \bullet 3^{\min(1,2)} \bullet 5^{\min(2,3)} \bullet 7^{\min(1,1)} \bullet 11^{\min(1,0)} \bullet 13^{\min(0,1)}$$

Euclid's Algorithm

- GCD(x, y) = GCD(y, x mod y)

```
int GCD(int a, int b){   /* a >= b,   b > 0 */
    int tmp;
    int x = a;
    int y = b;
    while (y > 0){
        tmp = x % y;
        x = y;
        y = tmp;
    }
    return x;
}
```

Multiplicative Inverse mod m

Suppose GCD(a, m) = 1

By Bézoit's Theorem, there exist integers s and t such that sa + tm = 1.

s mod m is the multiplicative inverse of a:

$$1 = (sa + tm) \bmod m = sa \bmod m$$

Induction proofs

$$\begin{array}{l} \quad P(0) \\ \quad \underline{\forall k\, (P(k) \rightarrow P(k+1))} \\ \therefore \forall n\, P(n) \end{array}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. $P(k) \rightarrow P(k+1)$ Direct Proof Rule
7. $\forall k\, (P(k) \rightarrow P(k+1))$ Intro $\forall$ from 2-6
8. $\forall n\, P(n)$ Induction Rule 1&7

Strong Induction

$$\begin{array}{l} \quad P(0) \\ \quad \underline{\forall k\, ((P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \rightarrow P(k+1))} \\ \therefore \forall n\, P(n) \end{array}$$

Recursive definitions of functions

- $F(0) = 0;\ F(n + 1) = F(n) + 1;$
- $G(0) = 1;\ G(n + 1) = 2 \times G(n);$
- $0! = 1;\ (n+1)! = (n+1) \times n!$
- $f_0 = 0; f_1 = 1; f_n = f_{n-1} + f_{n-2}$

Strings

- The set Σ^* of strings over the alphabet Σ is defined
  - Basis: $\lambda \in S$ (λ is the empty string)
  - Recursive: if $w \in \Sigma^*$, $x \in \Sigma$, then $wx \in \Sigma^*$
- Palindromes: strings that are the same backwards and forwards.
  - Basis: λ is a palindrome and any $a \in \Sigma$ is a palindrome
  - If *p* is a palindrome then a*p*a is a palindrome for every $a \in \Sigma$

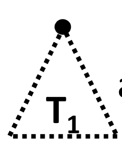
Function definitions on recursively defined sets

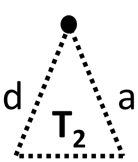
Len(λ) = 0;
Len(wx) = 1 + Len(w); for $w \in \Sigma^*$, $x \in \Sigma$

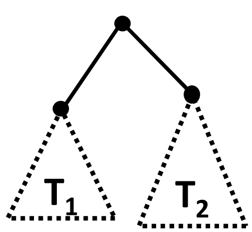
Concat(w, λ) = w for $w \in \Sigma^*$
Concat(w_1,w_2x) = Concat(w_1,w_2)x for w_1, w_2 in Σ^*, $x \in \Sigma$

Prove:
Len(Concat(x,y))=Len(x)+Len(y) for all strings x and y

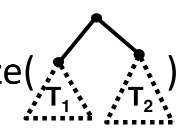
Rooted Binary trees

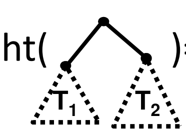
- Basis: ● is a rooted binary tree
- Recursive Step: If 

and 

are rooted binary trees then so is: 

Functions defined on rooted binary trees

- size(●)=1
- size(

) = 1+size(T_1)+size(T_2)
- height(●)=0
- height(

)=1+max{height(T_1),height(T_2)}

Prove:
For every rooted binary tree T, size(T) $\leq 2^{\mathbf{height(T)+1}}$ -1

Regular Expressions over Σ

- Each is a “pattern” that specifies a set of strings
- Basis:
  - $\emptyset$, λ are regular expressions
  - $\boldsymbol{a}$ is a regular expression for any $a \in \Sigma$
- Recursive step:
  - If **A** and **B** are regular expressions then so are:
    - $(\mathbf{A} \cup \mathbf{B})$
    - $(\mathbf{AB})$
    - $\mathbf{A}^*$

Regular Expressions

- $\boldsymbol{0^*}$
- $\boldsymbol{0^*1^*}$
- $(\boldsymbol{0} \cup \boldsymbol{1})^*$
- $(\boldsymbol{0^*1^*})^*$
- $(\boldsymbol{0} \cup \boldsymbol{1})^*\ \boldsymbol{0110}\ (\boldsymbol{0} \cup \boldsymbol{1})^*$
- $(\boldsymbol{0} \cup \boldsymbol{1})^*\ (\boldsymbol{0110} \cup \boldsymbol{100})(\boldsymbol{0} \cup \boldsymbol{1})^*$

Context-Free Grammars

- Example: $\mathbf{S} \rightarrow 0\mathbf{S}0 \mid 1\mathbf{S}1 \mid 0 \mid 1 \mid \lambda$

- Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \lambda$

Sample Context-Free Grammars

- Grammar for $\{0^n1^n : n \geq 0\}$ all strings with same # of 0’s and 1’s with all 0’s before 1’s.

- Example: $\mathbf{S} \rightarrow (\mathbf{S}) \mid \mathbf{SS} \mid \lambda$

Building in Precedence in Simple Arithmetic Expressions

- **E** – expression (start symbol)
- **T** – term **F** – factor **I** – identifier **N** - number

$\mathbf{E} \rightarrow \mathbf{T} \mid \mathbf{E}+\mathbf{T}$

$\mathbf{T} \rightarrow \mathbf{F} \mid \mathbf{F}*\mathbf{T}$

$\mathbf{F} \rightarrow (\mathbf{E}) \mid \mathbf{I} \mid \mathbf{N}$

$\mathbf{I} \rightarrow x \mid y \mid z$

$\mathbf{N} \rightarrow 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$

BNF for C

```
statement:
  ((identifier | "case" constant-expression | "default") ":")*
  (expression? ";" |
   block |
   "if" "(" expression ")" statement |
   "if" "(" expression ")" statement "else" statement |
   "switch" "(" expression ")" statement |
   "while" "(" expression ")" statement |
   "do" statement "while" "(" expression ")" ";" |
   "for" "(" expression? ";" expression? ";" expression? ")" statement |
   "goto" identifier ";" |
   "continue" ";" |
   "break" ";" |
   "return" expression? ";"
  )

block: "{" declaration* statement* "}"

expression:
  assignment-expression%

assignment-expression: (
    unary-expression (
      "=" | "*=" | "/=" | "%=" | "+=" | "-=" | "<<=" | ">>=" | "&=" |
      "^=" | "|="
    )
  )* conditional-expression

conditional-expression:
  logical-OR-expression ( "?" expression ":" conditional-expression )?
```


Definition of Relations

Let A and B be sets,
A binary relation from A to B is a subset of $A \times B$

Let A be a set,
A binary relation on A is a subset of $A \times A$

Let R be a relation on A

R is reflexive iff $(a,a) \in R$ for every $a \in A$

R is symmetric iff $(a,b) \in R$ implies $(b, a) \in R$

R is antisymmetric iff $(a,b) \in R$ and $a \neq b$ implies $(b,a) \notin R$

R is transitive iff $(a,b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

Combining Relations

Let R be a relation from A to B
Let S be a relation from B to C
The composite of R and S, $S \circ R$ is the relation from A to C defined

$S \circ R = \{(a, c) \mid \exists\, b \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$

Relations

(a,b)∈ Parent: b is a parent of a
(a,b)∈ Sister: b is a sister of a
Aunt = Sister ° Parent
Grandparent = Parent ° Parent

$R^2 = R \circ R = \{(a, c) \mid \exists\, b$ such that $(a,b) \in R$ and $(b,c) \in R\}$

$R^0 = \{(a,a) \mid a \in A\}$
$R^1 = R$
$R^{n+1} = R^n \circ R$

$S \circ R = \{(a, c) \mid \exists\, b$ such that $(a,b) \in R$ and $(b,c) \in S\}$

n-ary relations

Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

Student_ID	Name	GPA
328012098	Knuth	4.00
481080220	Von Neuman	3.78
238082388	Russell	3.85
238001920	Einstein	2.11
1727017	Newton	3.61
348882811	Karp	3.98
2921938	Bernoulli	3.21
2921939	Bernoulli	3.54

Student_ID	Major
328012098	CS
481080220	CS
481080220	Mathematics
238082388	Philosophy
238001920	Physics
1727017	Mathematics
348882811	CS
1727017	Physics
2921938	Mathematics
2921939	Mathematics

Matrix representation for relations

Relation R on $A=\{a_1, \ldots a_p\}$

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, a_j) \in R, \\ 0 \text{ if } (a_i, a_j) \notin R. \end{cases}$$

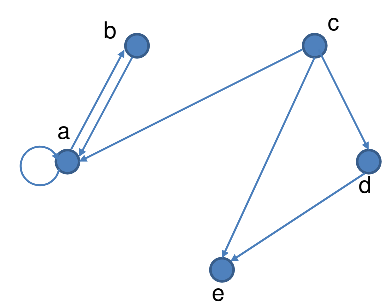
{(1, 1), (1, 2), (1, 4), (2,1), (2,3), (3,2), (3, 3) (4,2) (4,3)}

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Representation of relations

Directed Graph Representation (Digraph)

{(a, b), (a, a), (b, a), (c, a), (c, d), (c, e) (d, e) }



Paths in relations

Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$

(a,b) is in the transitive-reflexive closure of R if and only if there is a path from a to b. (Note: by definition, there is a path of length 0 from a to a.)

Finite state machines

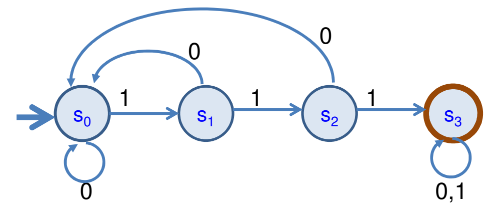
States

Transitions on inputs

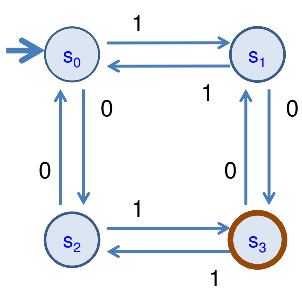
Start state and finals states

The language recognized by a machine is the set of strings that reach a final state

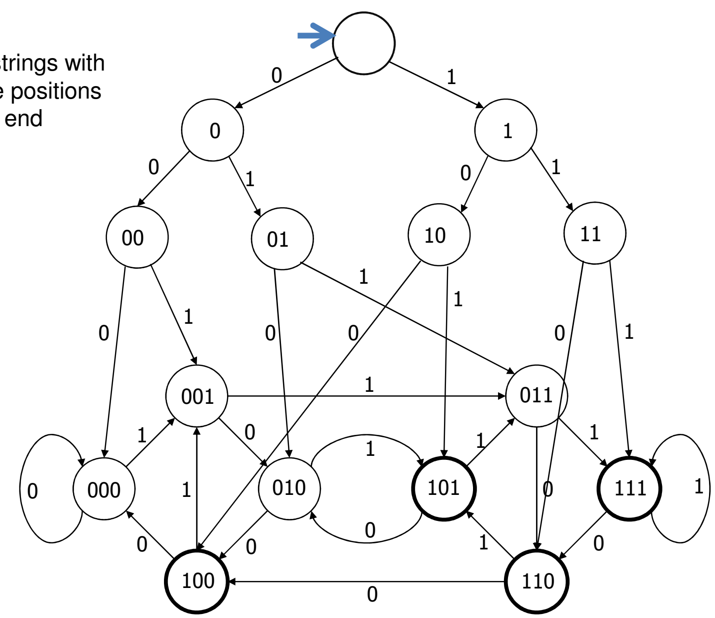
State	0	1
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_3
s_3	s_3	s_3



Accepts strings with an odd number of 1's and an odd number of 0's

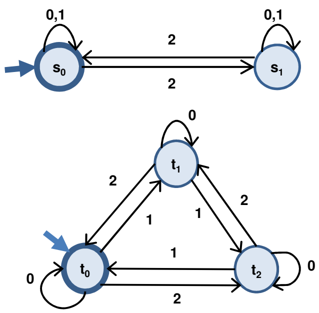


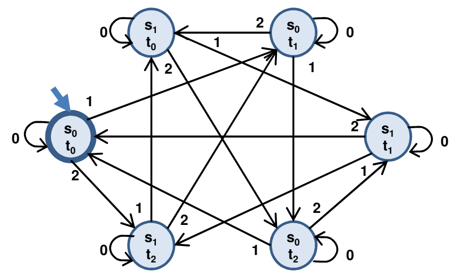
Accept strings with a 1 three positions from the end



Product construction

– Combining FSMs to check two properties at once
  - New states record states of both FSMs

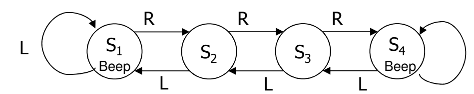




State machines with output

	Input		Output
State	L	R	
s_1	s_1	s_2	Beep
s_2	s_1	s_3	
s_3	s_2	s_4	
s_4	s_3	s_4	Beep

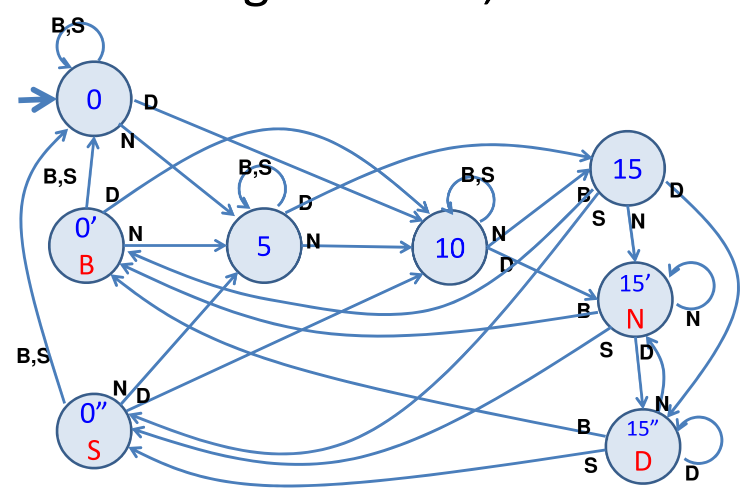
"Tug-of-war"



Vending Machine

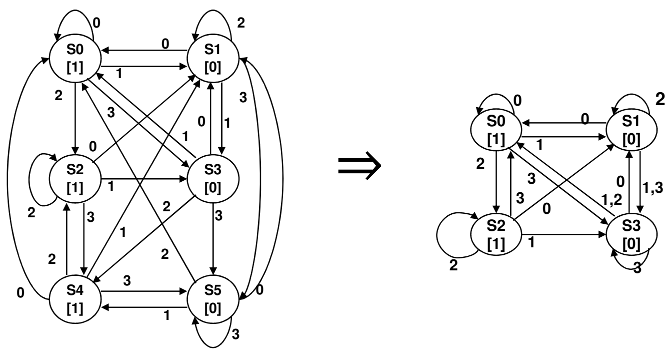
Enter 15 cents in dimes or nickels
Press S or B for a candy bar

Vending Machine, Final Version



State minimization

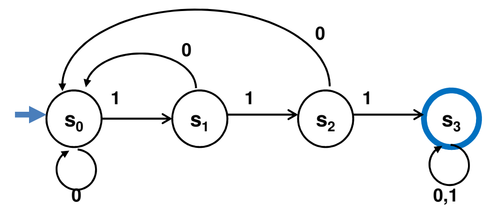
Finite State Machines with output at states



Another way to look at DFAs

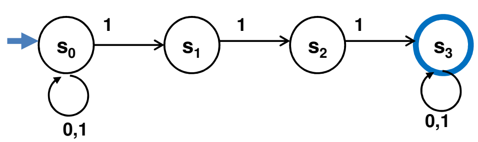
Definition: The label of a path in a DFA is the concatenation of all the labels on its edges in order

Lemma: x is in the language recognized by a DFA iff x labels a path from the start state to some final state

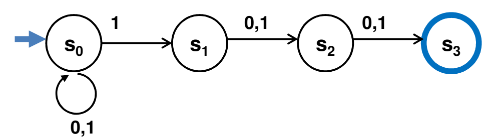


Nondeterministic Finite Automaton (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol - can have 0 or >1
  - Also can have edges labeled by empty string λ
- Definition: x is in the language recognized by an NFA iff x labels a path from the start state to some final state



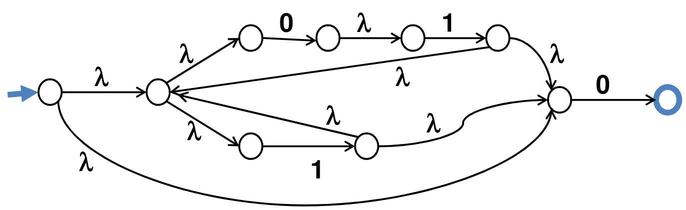
Nondeterministic Finite Automaton



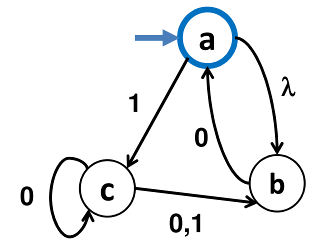
Accepts strings with a 1 three positions from the end of the string

Building a NFA from a regular expression

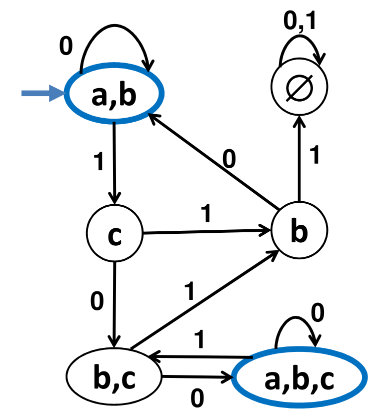
(01 ∪1)*0



NFA to DFA: Subset construction



NFA



DFA

The set B of binary palindromes cannot be recognized by any DFA

Consider the infinite set of strings

S={λ, 0, 00, 000, 0000, ...}

Claim: No two strings in S can end at the same state of any DFA for B, so no such DFA can exist

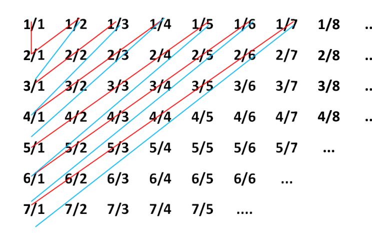
Proof: Suppose n≠m and 0^n and 0^m end at the same state p.

Since 0^n10^n is in B, following 10^n after state p must lead to a final state.

But then the DFA would accept 0^m10^n which is a contradiction

Cardinality

- A set S is *countable* iff we can write it as $S=\{s_1, s_2, s_3, ...\}$ indexed by ℕ
- Set of integers is countable
  - {0, 1, -1, 2, -2, 3, -3, 4, . . .}
- Set of rationals is countable
  - "dovetailing"



- Σ* is countable
  - {0,1}* = {0,1,00,01,10,11,000,001,010,011,100,101,...}
- Set of all (Java) programs is countable

The real numbers are not countable

- “diagonalization”

		1	2	3	4	5	6	7	8	9	...
D=	0.										
r_1	0.	5^{1}	0	0	0	0	0	0	0	...	...
r_2	0.	3	3^{5}	3	3	3	3	3	3	...	...
r_3	0.	1	4	2^{5}	8	5	7	1	4	...	...
r_4	0.	1	4	1	5^{1}	9	2	6	5	...	...
r_5	0.	1	2	1	2	2^{5}	1	2	2	...	...
r_6	0.	2	5	0	0	0	0^{5}	0	0	...	...
r_7	0.	7	1	8	2	8	1	8^{5}	2	...	...
r_8	0.	6	1	8	0	3	3	9	4^{5}	...	...
...	....	...	....	....	...	...	...	...	...	......	

General models of computation

Control structures with infinite storage
Many models
- Turing machines
- Functional
- Recursion
- Java programs

Church-Turing Thesis
Any reasonable model of computation that includes all possible algorithms is equivalent in power to a Turing machine

What is a Turing Machine?

Halting Problem

- **Given:** the code of a program **P** and an input **x** for **P**, i.e. given **(<P>,x)**
- **Output:** **1** if **P** halts on input **x**
  0 if **P** does not halt on input **x**

Theorem (Turing): There is no program that solves the halting problem
“The halting problem is undecidable”

Suppose H(<P>,x) solves the Halting problem

Does D halt on input <D>?

Function D(x):
```
  if H(x,x)=1 then
      while (true); /* loop forever */
  else
      no-op; /* do nothing and halt */
  endif
```

D halts on input <D>

$\Leftrightarrow$ H outputs 1 on input (<D>,<D>)

[since H solves the halting problem and so H(<D>,x) outputs 1 iff D halts on input x]

$\Leftrightarrow$ D runs forever on input <D>

[since D goes into an infinite loop on x iff H(x,x)=1]

Does a program have a divide by 0 error?

Input: A program <P> and an input string x

Output: 1 if P has a divide by 0 error on input x

0 otherwise

Claim: The divide by zero problem is undecidable

Program equivalence

Input: the codes of two programs, <P> and <Q>

Output: 1 if P produces the same output as Q does on every input

0 otherwise

Claim: The equivalent program problem is undecidable

Teaching evaluation

- Please answer the questions on both sides of the form. This includes the ABET questions on the back